

# On Efficiency of the English Auction<sup>★</sup>

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## Abstract

We study efficiency properties of the irrevocable exit English auction in a setting with interdependent values. Maskin [8] shows that the pairwise single-crossing condition is necessary for efficiency of the English auction with two bidders and suggests that it is also a sufficient condition. This paper clarifies and extends Maskin's results to the case of  $N$  bidders. We introduce the *generalized single crossing*—a fairly intuitive extension of the pairwise single-crossing—and prove that it is *essentially a necessary and sufficient* condition for the existence of an efficient equilibrium in the  $N$ -bidder English auction.

*Key words:* English auction, efficient auction, ex post equilibrium, single-crossing, interdependent values.

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## 1 Introduction

How to sell a good efficiently—to the buyer who values it the most—is one of the main questions of the theory of auctions. The task becomes harder as the informational environment gets more complex. When the valuations of the buyers are asymmetric and depend on the private information of the others the set of efficient mechanisms is quite limited. Among these is the open ascending price, or English, auction. It is typically modelled as the irrevocable exit clock auction, and this model is known to possess an efficient equilibrium when

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value functions satisfy certain conditions. What is the minimal (necessary and sufficient) condition for efficiency of the English auction is a long-standing problem. This paper provides a solution.

In a classic paper, Milgrom and Weber [9] introduce the irrevocable exit model of the English auction. They show that in the setting with symmetric interdependent values the English auction has an efficient equilibrium, and, if signals (the bidders' private information) are affiliated, it generates higher revenues to the seller than other common auction forms. Maskin [8] suggests that the pairwise single-crossing condition is necessary for efficiency of the asymmetric English auction, and shows that it is a sufficient condition when there are two bidders. Perry and Reny [11] provide an example with three bidders, where the pairwise single-crossing is satisfied and no efficient equilibrium exists. Krishna [7] presents a pair of sufficient conditions for efficiency of the  $N$ -bidder English auction—an average-crossing and a cyclical-crossing conditions.<sup>1</sup>

We introduce the *generalized single-crossing (GSC)* condition which is a natural extension of the pairwise single-crossing to multiple bidders. The pairwise single-crossing requires: if, starting from a signal profile where the values of two bidders are equal and maximal, we slightly increase the signal of one of the bidders, then her value remains the highest. This implies that a bidder's own private information affects her own valuation more than the valuations of her competitors. Our condition requires: if, starting from a signal profile where the values of a group of bidders are equal and maximal, we slightly increase the signals of a subset of the group, no bidder outside of the subset can attain the value higher than the maximal value attained among the bidders from the subset. The generalized single-crossing both implies the pairwise single-crossing and reduces to it in the case of two bidders.

Two main results of this paper are the *necessity*: if the generalized single-crossing condition is violated, then, except in three special cases, no efficient equilibrium in the  $N$ -bidder English auction exists; and the *sufficiency*: if value functions satisfy the generalized single-crossing condition, then there exists an efficient ex post equilibrium in the  $N$ -bidder English auction.

Two of the exceptions are related to the properties of the value functions on the boundary of the signals' domain and arise in the two-bidder English auction as well.<sup>2</sup> Examples 3 and 4 below show that an efficient equilibrium

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<sup>1</sup> The average-crossing condition requires that, if starting from a signal profile where the values of several bidders are equal and maximal, the signal of one of them is increased, the corresponding increments to the values of the others are lower than the average increment. The cyclical-crossing requires that the increments to the values are ranked in the pre-specified cyclical order—the effect on the own value is the largest and decreases for each subsequent bidder in the cycle.

<sup>2</sup> The third exception requires a peculiar combination of the value functions and the

may exist even if the pairwise single-crossing is violated in the interior of the signals' domain. This clarifies the conventional wisdom that the single-crossing condition is necessary for efficiency of any mechanism.<sup>3</sup>

The English auction is not the only efficient mechanism in the interdependent values setting, and GSC is not the weakest condition for efficiency. What makes the English auction so special, aside from its widespread use, is the strategic simplicity, transparent set of rules, and ease of conducting. In the English auction, even if the values are interdependent, the strategy in the efficient equilibrium is nothing but "...drop out when the price reaches what you believe your value is."

In contrast, the "contingent bid" mechanism of Dasgupta and Maskin [2] requires each buyer to submit a price she is willing to pay given the realized values of the others—a  $(N - 1)$ -variable function. This auction is efficient if the pairwise single-crossing holds. Utilizing the fact that two-bidder sealed bid and ascending price auctions are efficient, Perry and Reny [10] and Perry and Reny [11] design two elegant mechanisms which incorporate a concept of "directed bids" —every buyer bids against every other buyer, thus managing  $N - 1$  bids simultaneously. These auctions require the strong form of the pairwise single-crossing for efficiency.<sup>4</sup> All of these mechanisms are remarkable constructions, designed to allocate multiple units efficiently. In their single unit version, however, they are significantly more complex than the English auction. A generalized Vickrey auction is efficient if the pairwise single-crossing holds. In it, the bidders report their signals to the auctioneer, who assigns the object to the bidder with the highest value based on the reports and prescribes the payment. While the mechanism is simple, the auctioneer has to know what all the bidders commonly know, and the bidders have to trust the auctioneer to run the mechanism correctly.

Constructing an efficient equilibrium in the English auction is based on solving a vector-system  $p \cdot \mathbf{1} = \mathbf{V}(\mathbf{s})$  (equating the price and the valuations of the bidders, see Section 3 for details). An efficient equilibrium exists if the solution  $\mathbf{s}(p)$  is non-decreasing, and the generalized single-crossing is the exact condition guaranteeing that. As such, GSC can be exploited directly or

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signals' distribution. It may arise in settings with three or more bidders, in which some bidders drop out in a manner that does not reveal their signals. In essence, only if (1) ex post efficiency is not affected by the information of these bidders, and (2) the distribution of signals and the valuations are such that the expected values of the remaining bidders are *decreasing* in own signal, an efficient equilibrium may exist despite the violation of the pairwise single-crossing.

<sup>3</sup> See, for instance, Maskin [8], pp. 127-128; Dasgupta and Maskin [2], pp. 348-349.

<sup>4</sup> The pairwise single-crossing has to be satisfied for any pair of bidders with equal values, not only when their values are maximal. Thus, the strong pairwise single-crossing and GSC are not comparable.

with appropriate modifications in the analysis of any economic system described by a vector-system  $\mathbf{p} = \mathbf{f}(\mathbf{x})$  and that requires the monotonicity of  $\mathbf{x}(\mathbf{p})$ . Echenique and Manelli [4] highlight a connection between the theories of auctions and trade—where prices or quantities of final goods as functions of factor prices or factor demands are examples of such systems. They provide a *dominant-effect property (DEP)*, use it to generalize the Stolper-Samuelson Theorem, and show that it is also a sufficient condition for the existence of an efficient equilibrium in the  $N$ -bidder English auction. In essence, DEP is a global extension of GSC; it effectively requires GSC to be satisfied for any starting profile and for all bidders (not only for the subgroup whose valuations are equal and maximal) and all possible increases of signals (not only infinitesimal). As such, DEP is a much stronger sufficient condition for efficiency, and is far from being necessary.<sup>5</sup>

Izmalkov [5] proposes an alternative model of the English auction. In this model the bidders are allowed to reenter—become active again after they dropped out. The English auction with reentry is efficient under the conditions that are weaker than GSC.<sup>6</sup> At the same time the possibility of reentry substantially enriches the strategy space and provides opportunities to exchange messages, which, potentially, may allow bidders to coordinate on a collusive outcome. In contrast, the irrevocable-exit English auction is robust to collusion within the auction. The fact that the exits are irrevocable implies that the bidders cannot coordinate their actions: the only way a bidder can send a message is by exiting, which makes winning impossible.

Achieving full efficiency in more general interdependent values settings than considered in this paper has been shown to be problematic. Dasgupta and Maskin [2] and Jehiel and Moldovanu [6] prove that the first-best efficiency is generically impossible when the signals are multidimensional. Bergemann and Valimaki [1] show that ex ante information acquisition is inefficient when the values are strictly interdependent. The theoretical and practical values of known efficient mechanisms, including the English auction, are not diminished in such settings, however. It is likely that these mechanisms achieve the second-best or constrained efficient allocations—maximizing social welfare subject to all incentive constraints. (Dasgupta and Maskin [2] argue that their mechanism is constrained efficient.)

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<sup>5</sup> For two bidders, for instance, DEP reduces to the strong version of the single-crossing condition. For auctions with more than two bidders, DEP is not directly comparable to other known sufficient conditions. One of the objectives of Echenique and Manelli [4] is to dispense with differentiability conditions on the value functions, which DEP does not require explicitly. Yet, the monotonicity and continuity of the value functions imply partial differentiability along the equilibrium path of the efficient equilibrium construction, which, by itself suffices. Dubra [3] further weakens DEP and monotonicity conditions.

<sup>6</sup> GSC would imply that no reentry happens in an efficient equilibrium.

## 2 Preliminaries

### 2.1 The model

There is a single indivisible good to be auctioned among a set  $\mathcal{N} = \{1, 2, \dots, N\}$  of bidders. Prior to the auction each bidder  $j$  receives a real valued signal  $s_j \in [0, 1]$ . Signal  $s_j$  is bidder  $j$ 's private information. Signals are distributed according to a joint density function  $f(\mathbf{s})$ , where profile  $\mathbf{s} = (s_1, s_2, \dots, s_N)$  represents signals of all the bidders. It is assumed that  $f$  has full support and is strictly positive on the interior of it.

If the realized signals are  $\mathbf{s}$ , the value of the object to bidder  $j$  is  $V_j(\mathbf{s})$ —it depends potentially on the information obtained by the other bidders. The sale of an oil track is a typical example of such an environment—a firm's estimate of the worth of the track may depend on the results of the “off-site” drilling conducted by a rival firm that owns an adjacent track, see Porter [12].

Value functions  $\mathbf{V} = (V_1, V_2, \dots, V_N)$  are assumed to have the following properties. For any  $j$ , and any  $i \neq j$ :  $V_j(\mathbf{0}) = 0$ ;  $V_j(\mathbf{1}) < \infty$ ;  $V_j$  is continuously differentiable,  $V_j \in C^1([0, 1]^N)$ ; and  $\frac{\partial V_j}{\partial s_j} > 0$ . Value functions  $V_j$  for all  $j$  and distribution  $f(\mathbf{s})$  are commonly known among the bidders.<sup>7</sup>

We denote by  $\mathbf{s}_{\mathcal{A}} = (s_j)_{j \in \mathcal{A}}$  the signal profile of the bidders from a subset  $\mathcal{A} \subset \mathcal{N}$ , and by  $\mathbf{s}_{-\mathcal{A}}$  the signal profile of the bidders from  $\mathcal{N} \setminus \mathcal{A}$ .<sup>8</sup> By  $\mu_k(\mathbf{u})$  we denote the derivative of  $V_k$  in direction  $\mathbf{u}$ ,  $\mu_k(\mathbf{u}) = \mathbf{u} \cdot \nabla V_k(\mathbf{s}) = \sum_{i=1}^N u_i \frac{\partial V_k}{\partial s_i}$ , where  $\nabla V_k(\mathbf{s}) = \left( \frac{\partial V_k}{\partial s_1}, \frac{\partial V_k}{\partial s_2}, \dots, \frac{\partial V_k}{\partial s_N} \right)$  is the gradient of  $V_k(\mathbf{s})$ .

**Definition 1** For a given profile of signals  $\mathbf{s}$ , the *winners' circle*  $\mathcal{I}(\mathbf{s})$  is the set of bidders with the highest values at  $\mathbf{s}$ . Formally,

$$j \in \mathcal{I}(\mathbf{s}) \iff V_j(\mathbf{s}) = \max_{i \in \mathcal{N}} V_i(\mathbf{s}). \quad (1)$$

<sup>7</sup> These are fairly standard assumptions. We have normalized the supports of bidders' signals to  $[0, 1]$ , the analysis can be easily extended to the unbounded supports as long as the valuations of all the bidders are bounded. No assumption on the distribution of signals is needed for the sufficiency proposition. The necessity proposition requires that signals have full support and for all  $j$  and  $s_j$  any subset of the other bidders' signals of positive measure has positive probability. The assumption of  $f(\mathbf{s}) > 0$  is the simplest one guaranteeing these two properties. (For instance, if  $f$  is degenerate so that  $\mathbf{s}$  is commonly known, then  $\mathbf{V}(\mathbf{s})$  is known too and an efficient allocation can be achieved no matter what are the properties of the value functions.)

<sup>8</sup> We denote vectors and sets by bold and calligraphic letters correspondingly;  $\mathbf{a} \gg \mathbf{b}$  ( $\mathbf{a} \geq \mathbf{b}$ ) denotes that  $a_i > b_i$  ( $a_i \geq b_i$ ) in every component.

We require value functions to be *regular*: at every  $\mathbf{s}$  for any subset of bidders  $\mathcal{J} \subset \mathcal{I}(\mathbf{s})$  it is assumed that  $\det DV_{\mathcal{J}} \neq 0$ , where  $DV_{\mathcal{J}} = \left( \frac{\partial V_i(\mathbf{s})}{\partial s_j} \right)_{i,j \in \mathcal{J}}$  is the matrix of partial derivatives (Jacobian).

So far, we had made no assumptions on how a bidder's signal affects other bidders values. We do allow for such effects to be negative but not arbitrarily large. We require that at any  $\mathbf{s}$ , for any subset  $\mathcal{J} \subset \mathcal{I}(\mathbf{s})$ , and any direction  $\mathbf{u} = (\mathbf{u}_{\mathcal{J}}, \mathbf{u}_{-\mathcal{J}})$ , such that  $\mathbf{u}_{\mathcal{J}} \gg \mathbf{0}$  and  $\mathbf{u}_{-\mathcal{J}} = \mathbf{0}$ , there exists  $j \in \mathcal{J}$  with  $\mu_j(\mathbf{u}) > 0$ . That is, if all  $\mathbf{s}_{\mathcal{J}}$  are increased, there exists at least one bidder in  $\mathcal{J}$  whose value is increased.<sup>9</sup>

This assumption is weaker than any of the comparable restrictions of the existing literature. In many papers, including Milgrom and Weber [9], it is assumed that an increase in a bidder's signal has a non-negative effect on the other bidders values. Krishna [7] requires that such an increase, while possibly negatively affecting some bidders, has a positive total effect, which together with the average crossing condition implies the assumption imposed here.<sup>10</sup>

## 2.2 Generalized Single Crossing

**Definition 2** *The **generalized single-crossing (GSC)** condition holds if at any  $\mathbf{s}$  with  $\#\mathcal{I}(\mathbf{s}) \geq 2$ , for any subset  $\mathcal{A} \subset \mathcal{I}(\mathbf{s})$  and bidder  $k \in \mathcal{I}(\mathbf{s}) \setminus \mathcal{A}$ , and any direction  $\mathbf{u}$ , such that  $u_j > 0$  for all  $j \in \mathcal{A}$  and  $u_j = 0$  for all  $j \notin \mathcal{A}$ ,*

$$\mu_k(\mathbf{u}) \leq \max_{j \in \mathcal{A}} \{\mu_j(\mathbf{u})\}. \quad (2)$$

In words, select any group  $\mathcal{A}$  of bidders from  $\mathcal{I}(\mathbf{s})$ —bidders who have equal and

<sup>9</sup> To see what happens if the condition does not hold, consider the following value functions:  $V_1 = s_1 - 3s_2$  and  $V_2 = s_2 - 3s_1$ , and suppose that  $s_1$  and  $s_2$  are *iid*. Here, an efficient allocation is incentive compatible since, as we shall see, the single-crossing is satisfied, but no efficient equilibrium exists that is also individually rational. Indeed, it is efficient to allocate the good to  $j$  when  $s_j > s_{-j}$ , and so any pair of symmetric increasing bidding strategies produces an efficient allocation. But even if bidders always pay 0, the expected payoff from winning,  $E(s_j - 3s_{-j} | s_j > s_{-j})$ , is negative for some and possibly even for all  $s_j$ ! (This is so, for instance, when the signals are uniformly distributed.) In the English auction, even if  $V_j(\mathbf{0})$  is re-normalized a sufficiently high value so that the value of any bidder at any signal profile is positive, the symmetric increasing bidding strategies required for efficiency cannot be supported in equilibrium. It can be shown that each bidder would rather “pretend” to have a lower signal, since by exiting earlier she would avoid winning against opponents with “similar” signals and paying too high of a price.

<sup>10</sup> Note that the cyclical crossing condition is sufficient when the values are non-decreasing in the others bidders' signals (see Krishna [7]).

maximal values—and increase their signals. GSC requires that the increments to the values of bidders from  $\mathcal{I}(\mathbf{s}) \setminus \mathcal{A}$  are at most as high as the highest increment among the bidders from  $\mathcal{A}$ . Or, stated differently, at least one bidder from  $\mathcal{A}$  should be in the resulting winners' circle. *Single-crossing condition (SC)* is GSC required for subsets  $\mathcal{A}$  consisting of one bidder.

GSC is *violated* at the signal profile  $\mathbf{s}$  for subset  $\mathcal{A} \subset \mathcal{I}(\mathbf{s})$  and  $k \in \mathcal{I}(\mathbf{s}) \setminus \mathcal{A}$  if there exists vector  $\mathbf{u}$ , with  $u_j > 0$  for all  $j \in \mathcal{A}$ ,  $u_j = 0$  for all  $j \notin \mathcal{A}$ , such that  $\mu_k(\mathbf{u}) > \max_{i \in \mathcal{A}} \{\mu_i(\mathbf{u})\}$ . In what follows, whenever we say that GSC is violated it means that there exist such  $\mathbf{s}$ ,  $\mathcal{A}$ , and  $k$ .

GSC is *irreducibly violated* at  $\mathbf{s}$  for subset  $\mathcal{A}$  and bidder  $k$  if: (i)  $\mathcal{A} \cup k = \mathcal{I}(\mathbf{s})$ ; and (ii) there exists an open neighborhood  $U^{\mathbf{s}}$  of  $\mathbf{s}$ , so that for all  $\mathbf{s}' \in U^{\mathbf{s}}$ , GSC is satisfied at  $\mathbf{s}'$  for any subset  $\mathcal{B} \subset \mathcal{I}(\mathbf{s})$  with  $\#\mathcal{B} < \#\mathcal{A}$  and bidder  $l \in \mathcal{I}(\mathbf{s}) \setminus \mathcal{B}$ . Such a subset  $\mathcal{A}$  is called *irreducible*.

In essence, if GSC is violated irreducibly for  $\mathcal{A}$  and  $k$ , then one cannot (locally) find a smaller number of bidders for which GSC is violated. Our necessity proposition uses this notion. The following Lemma, proven in Appendix A.2, states that if GSC is violated at an interior  $\mathbf{s}$ , one can find a close-by  $\mathbf{s}'$ , at which GSC is irreducibly violated, although possibly for different  $\mathcal{A}$  and  $k$ .

**Lemma 1** *Suppose GSC is violated at an interior profile  $\mathbf{s}$ . Then, for all  $\varepsilon > 0$ , there exists  $\mathbf{s}'$ ,  $|\mathbf{s}' - \mathbf{s}| < \varepsilon$ , such that GSC is irreducibly violated at  $\mathbf{s}'$ .*

### 2.3 The English Auction

Following Milgrom and Weber [9], we consider a model that became the standard model for the analysis of English auctions. Specifically, the price of the object rises continuously, and bidders indicate whether they are willing to buy the object at that price or not. A bidder who is willing to buy at the current price is said to be an active bidder. At a price of 0 all the bidders are active, and, as the price rises, bidders can choose to drop out of the auction. The decision to drop out is both public and irrevocable. Thus, if bidder  $j$  drops out at a price  $p_j$ , both her identity and the exiting price  $p_j$  are observed by all the bidders. Furthermore, once bidder  $j$  drops out she cannot “reenter” the auction at a higher price. The auction ends when at most one bidder remains active. The clock stops, the only remaining bidder is the winner. If no bidders remain active the winner is chosen at random among those who exited last. The winner is obliged to pay the price shown on the clock.<sup>11</sup>

<sup>11</sup> We should complete the description of the game by specifying the outcome in the case where two or more bidders decide to remain active forever, and an auction continues indefinitely. In this case we assign to every such bidder a payoff of  $-\infty$

Since all drop-out decisions are public, the *public history*  $H(p)$  can be effectively summarized as a sequence of prices at which bidders, inactive at  $p$ , have exited,  $H(p) = \mathbf{p}_{-\mathcal{M}}$ , where  $\mathcal{M}$  is the set of active bidders just before  $p$ . If no bidder exits at  $p \in [p', p'']$ , then  $H(p') = H(p'')$ . Denote with  $\bar{H}(p)$  the public history  $H(p)$  together with all exits that happen at  $p$ . Therefore, if  $\bar{H}(p) \neq H(p)$ , then there exists a bidder who exited at  $p$ . All the bidders are assumed to be active just before the clock starts at  $p = 0$ , so  $H(0) = \emptyset$ .

In the English auction a bidder's strategy determines the price at which she would drop out given the public history and if no other bidder drops out first. Formally, following Krishna [7], bidder  $j$ 's bidding strategy is a collection of functions  $\beta_j^{\mathcal{M}} : [0, 1] \times \mathbb{R}_+^{N-M} \rightarrow \mathbb{R}_+$ , for each  $\mathcal{M} \subset \mathcal{N}$ , with  $j \in \mathcal{M}$  and  $M = \#\mathcal{M} > 1$ . Function  $\beta_j^{\mathcal{M}}$  determines the price  $\beta_j^{\mathcal{M}}(s_j; H(p))$  at which bidder  $j$  with signal  $s_j$  will drop out when the set of active bidders is  $\mathcal{M}$  and the bidders  $\mathcal{N} \setminus \mathcal{M}$  dropped out at prices  $H(p) = \mathbf{p}_{-\mathcal{M}} = \{p_j\}_{j \in \mathcal{N} \setminus \mathcal{M}}$ . If  $\beta_j^{\mathcal{M}}(s_j; H(p)) \leq p$  at some  $p$ , the strategy says that bidder  $j$  must exit at  $p$ . If active bidders are able to infer true signals  $\mathbf{s}_{-\mathcal{M}}$  of inactive bidders from their exit prices  $\mathbf{p}_{-\mathcal{M}}$ , the strategies can be equivalently written as  $\beta_j^{\mathcal{M}}(s_j; \mathbf{s}_{-\mathcal{M}})$ . Since public history uniquely determines the set of active bidders  $\mathcal{M}$ , we simply use  $\beta_j(s_j; H(p))$  in place of  $\beta_j^{\mathcal{M}}(s_j; H(p))$  in the rest of the paper.

The equilibrium concept we use is a Bayesian-Nash equilibrium. The equilibrium we present in Section 3 is also ex-post and efficient.

**Definition 3** *An **ex-post equilibrium** is a Bayesian-Nash equilibrium  $\beta$  that remains a Nash equilibrium even if the signals  $(s_1, s_2, \dots, s_N)$  are commonly known. An equilibrium is **efficient** if the object is allocated to the bidder with the highest value at every realization of signals  $(s_1, s_2, \dots, s_N)$ .*

The following is a known example in which the English auction is not efficient.

**Example 1 (Perry and Reny)** *Consider the following value functions*

$$\begin{aligned} V_1 &= s_1 + s_2 s_3, \\ V_2 &= \frac{1}{2} s_1 + s_2, \\ V_3 &= s_3. \end{aligned}$$

*There exists no efficient equilibrium in the English auction.*

Perry and Reny [11] prove that no efficient equilibrium exists. Observe that GSC is violated at  $\mathbf{s} = (.3, .6, .75)$  for  $\mathcal{A} = \{2, 3\}$ , bidder 1, and vector  $\mathbf{u} = (0, 1, 1)$ . Indeed,  $\mu_2(\mathbf{u}) = \mu_3(\mathbf{u}) = 1$ , while  $\mu_1(\mathbf{u}) = \frac{\partial V_1(\mathbf{s})}{\partial s_2} + \frac{\partial V_1(\mathbf{s})}{\partial s_3} = 1.35 > 1$ .

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(alternatively, it suffices to set that the object is not allocated).

### 3 Sufficiency

In this section we show that GSC is sufficient for the existence of an efficient equilibrium in the  $N$ -bidder English auction.<sup>12</sup>

**Proposition 1 (Sufficiency)** *Suppose value functions satisfy GSC. Then, there exists an efficient ex post equilibrium in the  $N$ -bidder English auction.*

The key component of the proposed efficient equilibrium is existence of a weakly-increasing profile of *inferences*, or *inverse bidding functions*,  $\sigma(p, H(p))$ , computed at any  $p$  for any given public history  $H(p) = \mathbf{p}_{-\mathcal{M}}$ , where  $\mathcal{M}$  is the set of active bidders. Then, the equilibrium strategy of each active bidder  $j$  prescribes to remain active as long as  $\sigma_j(p, H(p))$  is lower than  $s_j$ , and to exit at the lowest  $p_j$  at which  $s_j = \sigma_j(p_j, H(p))$ .

Formally, we define the (suggested equilibrium) strategies as follows. Suppose there exists a profile of functions  $\sigma(p, H(p))$ , such that, letting  $\mathcal{M}$  be the set of active bidders given  $H(p)$ , for any  $p \geq \max_{i \notin \mathcal{M}} p_i$ :

- (1) for any (inactive) bidder  $i \notin \mathcal{M}$ ,  $\sigma_i(p) = \sigma_i(p_i, H(p_i))$ ,<sup>13</sup> that is,  $\sigma_i(p)$  is fixed after bidder  $i$  exits at  $p_i$ ;
- (2) for any bidder  $j \in \mathcal{M}$ ,  $\sigma_j(p) \in [0, 1]$  solves  $V_j(\sigma_j(p), \sigma_{-j}(p)) = p$  if such a solution exists with  $\sigma_j(p) \leq 1$ , else  $\sigma_j(p) = 1$  and  $V_j(\sigma_j(p), \sigma_{-j}(p)) < p$ .

That is, for all active bidders,  $\sigma_{\mathcal{M}}(p)$  are determined simultaneously as a solution to

$$\begin{aligned} \mathbf{V}_{\mathcal{M}}(\sigma_{\mathcal{M}}(p), \sigma_{-\mathcal{M}}(p)) &\leq p \mathbf{1}_{\mathcal{M}}, \quad \sigma_{\mathcal{M}}(p) \leq \mathbf{1}_{\mathcal{M}}, \\ \forall j : (V_j(\sigma(p)) - p)(\sigma_j(p) - 1) &= 0. \end{aligned} \tag{3}$$

Then, for bidder  $j \in \mathcal{M}$  strategy  $\beta_j^{\mathcal{M}} : (s_j, H(p)) \longrightarrow \mathbb{R}_+$  is

$$\beta_j^{\mathcal{M}}(s_j; \mathbf{p}_{-\mathcal{M}}) = \arg \min_p \{\sigma_j(p) \geq s_j\}. \tag{4}$$

Strategy  $\beta_j$  can be interpreted as follows. Given the public history  $H(p) = \mathbf{p}_{-\mathcal{M}}$ , an active bidder  $j$  is supposed to exit the auction at  $p_j = \beta_j^{\mathcal{M}}(s_j; \mathbf{p}_{-\mathcal{M}})$ ,

<sup>12</sup>We build upon the existing constructions. Milgrom and Weber [9] present an efficient equilibrium for the English auction with symmetric bidders; Maskin [8] extends it to the case of two asymmetric bidders, and Krishna [7] generalizes it to the case of  $N$  asymmetric bidders.

<sup>13</sup>To shorten notation we are omitting  $H(p)$  from the set of arguments, whenever the public history is explicitly mentioned or can be implied from the context.

provided no other bidder exits before. If the current price  $p < \beta_j^M(s_j; \mathbf{p}_{-\mathcal{M}})$ , bidder  $j$  is suggested to remain active; if  $p \geq \beta_j^M(s_j; \mathbf{p}_{-\mathcal{M}})$  bidder  $j$  is suggested to exit at  $p$ . Once bidder  $j$  exits at  $p_j$ , other bidders update the public history and, expecting bidder  $j$  to follow (4), infer  $s_j^* = \sigma_j(p_j)$ . If  $\sigma_j(\cdot)$  is non-decreasing the inferred  $s_j^*$  is unique and coincides with true signal  $s_j$ . The strategies can then be reformulated as functions of the own and inferred signals of inactive bidders,  $\beta_j^M(s_j; \mathbf{s}_{-\mathcal{M}}) = \beta_j^M(s_j; \mathbf{p}_{-\mathcal{M}})$ .

The proof of Proposition 1 is based on the following three lemmas.

**Lemma 2** *GSC is satisfied if and only if at any  $\mathbf{s}$  with  $\#\mathcal{I}(\mathbf{s}) \geq 2$ , for any subset of bidders  $\mathcal{A} \subset \mathcal{I}(\mathbf{s})$  and bidder  $k \in \mathcal{I}(\mathbf{s}) \setminus \mathcal{A}$ ,*

$$\mu_k(\mathbf{u}^{\mathcal{A}}) \leq 1, \quad (5)$$

where vector  $\mathbf{u}^{\mathcal{A}} = (\mathbf{u}_{\mathcal{A}}^{\mathcal{A}}, \mathbf{0}_{-\mathcal{A}})$  solves  $\boldsymbol{\mu}_{\mathcal{A}}(\mathbf{u}^{\mathcal{A}}) = \mathbf{1}$ .<sup>14</sup> In addition, if GSC is satisfied, then for any such  $\mathcal{A}$ ,  $\mathbf{u}_{\mathcal{A}}^{\mathcal{A}} \geq \mathbf{0}$ .

We will refer to  $\mathbf{u}^{\mathcal{A}}$  as *the equal increments vector* corresponding to subset  $\mathcal{A}$ .

Suppose that, starting at  $\mathbf{s}$ , the values of some of the bidders in the winners' circle,  $\mathcal{A} \subset \mathcal{I}(\mathbf{s})$ , are increased uniformly, while the signals of all the other bidders are fixed. For such change in values, the signals of the bidders from  $\mathcal{A}$  adjust in the direction  $\mathbf{u}_{\mathcal{A}}^{\mathcal{A}}$ . If GSC is satisfied, then by Lemma 2 no bidder from the winners' circle obtains the value higher than the value(s) of the bidders from  $\mathcal{A}$ , and the signals of the bidders from  $\mathcal{A}$  cannot decrease. Note also that Lemma 2 provides a relatively simple way of verifying GSC. Indeed, at any  $\mathbf{s}$  and for a given  $DV_{\mathcal{I}(\mathbf{s})}$ , it suffices to check whether linear inequalities (5) are jointly satisfied.

**Lemma 3** *Suppose GSC holds. Then there exist inferences  $\boldsymbol{\sigma}(p, H(p))$ , such that each  $\sigma_j(\cdot, H(p))$  is continuous and non-decreasing for any  $H(p)$ , and  $\sigma_j(p, \bar{H}(p)) = \sigma_j(p, H(p))$  for all  $p$  such that  $\bar{H}(p) \neq H(p)$ . For any active at  $H(p)$  bidder  $j$ ,  $\sigma_j(p, H(p)) < 1$  implies  $j \in \mathcal{I}(\boldsymbol{\sigma}(p))$ .*

This Lemma establishes existence of inferences satisfying (3). The proofs of Lemmata 2 and 3 are in Appendix A.1.

**Lemma 4** *Suppose GSC holds. Then  $\boldsymbol{\beta}$  defined by (4) constitute an efficient ex-post equilibrium in the  $N$ -bidder English auction.*

**Proof.** We first show that  $\boldsymbol{\beta}$  are well-defined. For any bidder  $j$ , arbitrarily fix exit prices of other bidders,  $\mathbf{p}_{-j}$ , possibly with  $p_i = \infty$  for some bidders. Then one can obtain  $\sigma_j(p)$  defined for any  $p \geq 0$  as  $\sigma_j(p) = \sigma_j(p, H(p))$ ,

<sup>14</sup> That is, for each  $i \in \mathcal{A}$ ,  $\mu_i(\mathbf{u}^{\mathcal{A}}) = 1$ . By regularity, vector  $\mathbf{u}^{\mathcal{A}}$  is uniquely defined as  $\mathbf{u}_{\mathcal{A}}^{\mathcal{A}} = (DV_{\mathcal{A}}(\mathbf{s}))^{-1} \cdot \mathbf{1}_{\mathcal{A}}$  and  $\mathbf{u}_{-\mathcal{A}}^{\mathcal{A}} = \mathbf{0}$ .

where  $H(p) = \cup_{p_i < p} \{p_i\}$ . Lemma 3 shows that  $\sigma_j(p)$  is continuous and non-decreasing for any given  $\mathbf{p}_{-j}$ . Therefore,  $p_j = \arg \min_p \{\sigma_j(p) \geq s_j\}$  is unique, so  $\beta_j(s_j; \cdot)$  is well defined.

Next, we show that when all the bidders follow strategies (4), the object is allocated efficiently. Suppose bidder  $j$  wins the object at price  $p^*$ . Then, for any  $i \neq j$ ,  $\sigma_i(p^*) = \sigma_i(p_i) = s_i$ , while  $\sigma_j(p^*) \leq s_j$ , and, according to Lemma 3,

$$V_j(\sigma_j(p^*), \mathbf{s}_{-j}) = \max_{i \neq j} V_i(\sigma_j(p^*), \mathbf{s}_{-j}) = p^*. \quad (6)$$

The pairwise single-crossing,  $\sigma_j(p^*) \leq s_j$ , and equation (6) imply that

$$V_j(\mathbf{s}) \geq \max_{i \neq j} V_i(\mathbf{s}) \geq p^*, \quad (7)$$

so bidder  $j$  is (one of) the bidder(s) with the highest value, and the price  $p^*$  that bidder  $j$  has to pay does not depend on the signal of bidder  $j$ .

Finally, we show that  $\beta$  form an ex-post equilibrium. Suppose every bidder other than bidder  $j$  follows the proposed strategy and bidder  $j$  deviates. The payoff of bidder  $j$  can change only if the deviation affects whether bidder  $j$  obtains the object. If bidder  $j$  wins the object as a result of the deviation, she has to pay  $p_j^* = \max_{i \neq j} V_i(\sigma_j(p_j^*), \mathbf{s}_{-j})$ . If bidder  $j$  is not the winner in the equilibrium, then  $\sigma_j(p_j^*) \geq s_j$  since  $\sigma_j(p)$  is non-decreasing, so  $V_j(\mathbf{s}) \leq p_j^*$ , and the deviation is not profitable. If, as a result of the deviation, bidder  $j$  is not the winner while she is in the equilibrium, she is possibly forfeiting positive profits according to (7). Thus, no profitable deviation exists. The above arguments are valid even if signals  $\mathbf{s}$  are commonly known, hence the presented equilibrium is ex-post.  $\square$

## 4 Necessity

In this section we establish that GSC is essentially necessary for the existence of an efficient equilibrium in the  $N$ -bidder English auction.

There are three exceptional situations, in which GSC is violated but an efficient equilibrium exists. We illustrate the exceptions on the examples in which SC is violated. Two of these examples involve two bidders only. All of the three exceptions extend to other efficient mechanisms, including the generalized Vickrey auction and the Dasgupta & Maskin mechanism. Thus, we demonstrate that SC is only essentially necessary for efficiency of the 2-bidder English auction and general  $N$ -bidder mechanisms.

We also provide an example where GSC is satisfied in the interior of the signals' domain but an efficient equilibrium does not exist.

#### 4.1 Examples

The following example illustrates that efficient equilibria may exist when SC is violated only on the boundary of the signals' domain.

**Example 2** Consider the following value functions

$$\begin{aligned} V_1 &= s_1 + s_2, \\ V_2 &= \frac{1}{3}s_1 + \frac{2}{3}s_2. \end{aligned}$$

*There exists an efficient ex post equilibrium in the English auction.*

At  $\mathbf{s} = \mathbf{0}$ ,  $V_1 = V_2$ , and the pairwise single-crossing is violated, while at any other  $\mathbf{s}$  it is vacuously satisfied since  $V_1 > V_2$ . Thus, in the interior bidder 1 always has the highest value. Strategies  $\beta_1(s_1) = \infty$ ,  $\beta_2(s_2) = s_2$  (bidder 1 never drops out first) form an ex post equilibrium, which is efficient.<sup>15</sup>  $\square$

In the next example an efficient equilibrium exists even when SC is violated in the interior of the signals' domain.

**Example 3** Consider the following value functions

$$\begin{aligned} V_1 &= Bs_1 + 2s_2 - \frac{1}{2}, \\ V_2 &= Bs_1 + s_2, \end{aligned}$$

*and the conditional density (with  $|A| < 4$ ):  $f(s_1|s_2) = 1 - A\left(s_1 - \frac{1}{2}\right)\left(s_2 - \frac{1}{2}\right)$ . If  $AB > 12$ , there exists an efficient equilibrium in the English auction.*<sup>16</sup>

<sup>15</sup>In this example bidder 1 always has the highest value. It is easy to construct similar examples with more than two bidders, so that any of the bidders may have (strictly) the highest value at some signal profiles. For instance, an efficient equilibrium exists if there are three bidders with value functions:  $V_1 = s_1 + \frac{2}{3}(s_2 + s_3)$ ,  $V_2 = s_2$ , and  $V_3 = s_3$ . Note that GSC is violated at  $\mathbf{s} = \mathbf{0}$ , and there is no other  $\mathbf{s}$  at which values of all three bidders are equal. The following strategies form an efficient ex post equilibrium. Bidders 2 and 3 drop out when the price reaches their private values. Bidder 1 waits (never drops out first) if both other bidders are active, and, after one of them, say bidder 2, exits at  $p_2$ , bidder 1 stays active until  $p = s_1 + \frac{2}{3}p_2$ .  
<sup>16</sup>The p.d.f.  $f(\mathbf{s})$  is strictly positive whenever marginal distribution of  $s_2$  is strictly positive ( $|A| < 4$  gives  $f(s_1|s_2) > 0$ ). The fact that  $\mathbf{V}(\mathbf{0}) \neq \mathbf{0}$  in the example is not important. We can redefine  $V_1(s_1, s_2)$  as follows:  $V_1 = Bs_1\left(1 - 4\left(\frac{1}{2} - s_2\right)^2\right) + 4Bs_1^2\left(\frac{1}{2} - s_2\right)^2 + 2s_2 - \frac{1}{2}$  for  $s_2 < \frac{1}{2}$  and  $V_1 = Bs_1 + 2s_2 - \frac{1}{2}$  for  $s_2 \geq \frac{1}{2}$ . The same argument applies.

Here, the pairwise single-crossing is violated at  $(s_1, \frac{1}{2})$  for any  $s_1$ . It is essential that  $\frac{\partial V_1(s_1, s_2)}{\partial s_1} = \frac{\partial V_2(s_1, s_2)}{\partial s_1}$  for all  $s_1$ , so that the identity of the bidder with the highest value is independent of  $s_1$  and is solely determined by  $s_2$ : it is efficient to allocate the good to bidder 1 when  $s_2 > \frac{1}{2}$  and to bidder 2 when  $s_2 < \frac{1}{2}$ . Also, the distribution of the signals is such that bidder 2 with a high signal is relatively pessimistic about bidder 1's signal and so has a low expected payoff from winning at a given price; while bidder 2 with a low signal is relatively optimistic and has a high expected payoff. This negative correlation is important, it is not possible to construct such an example with two bidders when the signals are independent.

The following pair of strategies constitutes an efficient equilibrium. Bidder 1 with any signal bids  $\beta_1 = \frac{B}{2} + \frac{1}{2}$ , while bidder 2 with  $s_2 \leq \frac{1}{2}$  bids  $\beta_2 = \infty$  and with  $s_2 > \frac{1}{2}$  bids  $\beta_2 = 0$ . Bidder 1 pays 0 whenever she wins. She cannot change her bid to increase her payoff. Bidder 2 with signal  $s_2 > \frac{1}{2}$  loses to  $\beta_1 = \frac{B}{2} + \frac{1}{2}$  and has a payoff of 0. She cannot improve since by staying active above  $\beta_1$  and winning she obtains

$$E(V_2 - \beta_1 | s_2) = \left(1 - \frac{1}{12}AB\right) \left(s_2 - \frac{1}{2}\right),$$

which is negative if  $AB > 12$ . Similarly, bidder 2 with  $s_2 \leq \frac{1}{2}$  wins and does not regret paying  $\frac{B}{2} + \frac{1}{2}$  since her expected payoff is positive.  $\square$

In the next example SC is also violated in the interior but the factors that make efficiency possible are different. They are: the presence and early exit of the third bidder in a way that does not reveal her signal,  $s_3$ ; the identical effect of  $s_3$  on the values of the other two bidders; and the negative correlation of  $s_3$  with the signals of the other bidders.

**Example 4** *Consider the following value functions*

$$V_1 = s_1 + 2s_2 + 4s_3,$$

$$V_2 = 2s_1 + s_2 + 4s_3,$$

$$V_3 = s_3.$$

*Suppose that  $s_3 = 1 - (s_1 + s_2)/2$ . There exists an efficient equilibrium in the English auction.*<sup>17</sup>

In this equilibrium bidder 3 exits at 0 for all  $s_3$ . Given this, the expected values of bidders 1 and 2 are  $W_1(s_1, s_2) = E_{s_3|s_1, s_2} V_1(s_1, s_2, s_3) = 4 - s_1$  and  $W_2(s_1, s_2) = 4 - s_2$ . Bidding strategies  $\beta_1(s_1; \cdot) = 4 - s_1$  and  $\beta_2(s_2; \cdot) = 4 - s_2$

<sup>17</sup>The fact that the signals distribution is degenerate is not consequential. Similar efficient equilibrium exists for any non-degenerate distribution that results in  $E(s_3|s_1, s_2) = A - (2A - 1)(s_1 + s_2)/2$  with  $A$  less than but close to 1.

form an efficient equilibrium. (Let  $t_1 = 1 - s_1$  and  $t_2 = 1 - s_2$ , then the expected values— $W_1 = 3 + t_1$  and  $W_2 = 3 + t_2$ —satisfy *SC* for these re-normalized signals, and  $\beta_i(t_i) = W_i(t_i)$  for  $i = 1, 2$  ensure efficiency.)<sup>18</sup>  $\square$

The next example illustrates that GSC being satisfied in the interior of the signals' domain is not sufficient for the existence of an efficient equilibrium.

**Example 5** *Consider the following value functions*

$$\begin{aligned} V_1 &= s_1 + \frac{3}{2}s_2s_3, \\ V_2 &= s_2 + \frac{3}{4}s_1, \\ V_3 &= s_3, \\ V_4 &= s_4 + \frac{1}{2}(s_1 + s_2 + s_3). \end{aligned}$$

*No efficient equilibrium exists in the English auction.*

At  $\mathbf{s} = \mathbf{0}$ , GSC is violated for subset  $\mathcal{A} = \{1, 2, 3\}$  and bidder 4 (for instance, for  $\mathbf{u} = \{1, 1, 1\frac{3}{4}, 0\}$ ), and satisfied at all interior profiles.<sup>19</sup> Suppose an efficient equilibrium exists, and consider the following bidders and signals:  $s_1 \in \{0, \frac{1}{10}, \frac{1}{5}\}$ ;  $s_2 \in \{\frac{1}{4}, \frac{3}{4}\}$ ;  $s_3 \in \{\frac{5}{16}, \frac{3}{4}\}$ ; and  $s_4 = 0$ . For every bidder  $j$  with signal  $s_j$ , her strategy  $\beta_j(s_j) = \beta_j(s_j, \emptyset)$  specifies the price at which  $j$  drops out if all the other bidders remain active. Who of these bidders has the lowest  $\beta_j(s_j)$ ? All of them cannot remain active forever if efficiency is to be achieved, thus, the lowest bid is well-defined. When  $(s_1, s_2, s_4) = (0, \frac{1}{4}, 0)$  bidder 3 with either signal,  $\frac{5}{16}$  or  $\frac{3}{4}$ , has the highest value, thus neither  $\beta_3(\frac{5}{16})$  nor  $\beta_3(\frac{3}{4})$  is the lowest. When  $(s_1, s_3, s_4) = (\frac{1}{5}, \frac{5}{16}, 0)$  bidder 2 with either signal has the highest value; when  $(s_1, s_2, s_3, s_4) = (\frac{1}{10}, \frac{1}{4}, \frac{5}{16}, 0)$  bidder 4 has the highest value; and when  $(s_2, s_3, s_4) = (\frac{3}{4}, \frac{3}{4}, 0)$  bidder 1 with either signal has the highest value. Thus none of the  $\beta_j(s_j)$  can be the lowest, which is a contradiction, and so an efficient equilibrium does not exist.

<sup>18</sup> While somewhat similar in construction, efficient equilibria of Examples 3 and 4 are quite different. First, active bidding of bidder 1 in Example 3 is crucial. Second, it can be shown that an efficient equilibrium with increasing bidding strategy of bidder 1 exists as well. In particular, for sufficiently small  $\varepsilon > 0$ ,  $\beta_1(s_1) = \varepsilon s_1 + \frac{B-\varepsilon}{2} + \frac{1}{2}$  and the same strategy of bidder 2 form an efficient equilibrium.

<sup>19</sup> In particular, there is no violation of GSC on subset  $\{2, 3\}$  and bidder 1 since whenever  $V_2 = V_3$ ,  $V_4 > V_3$ . To verify SC observe that if  $V_1 = V_2$  then  $\frac{1}{4}s_1 = s_2 - \frac{3}{2}s_2s_3$  and  $\frac{\partial V_2}{\partial s_2} - \frac{\partial V_1}{\partial s_2} = (1 - \frac{3}{2}s_3) = \frac{1}{4}\frac{s_1}{s_2}$  which is positive in the interior. Similarly,  $\frac{\partial V_3}{\partial s_3} - \frac{\partial V_1}{\partial s_3} > 0$  whenever  $V_1 = V_3$ .

## 4.2 Indistinguishable signals

In Example 4, an efficient equilibrium exists because there is a bidder whose information is *indistinguishable*, that is: (1) her signal does not affect who of the (other) bidders has the highest value; (2) her equilibrium strategy prescribes dropping out at the same price for any realization of the signal. Because a signal of such a bidder is not revealed by her observed action, the other bidders have to rely on their expected values in forming their strategic decisions. Without any restrictions on the distribution of signals, the expected values of the other bidders may have drastically different properties, such as being decreasing in their own signals, which, in turn, allows for an efficient equilibrium to exist despite a violation of GSC. We are going to impose a mild assumption jointly on the value functions and the distribution of signals, under which such an exception will not arise.

**Definition 4** A signal profile  $\mathbf{s}$  is **strictly competitive** for bidders  $j$  and  $k$  if  $\{j, k\} = \mathcal{I}(\mathbf{s})$  and  $\frac{\partial V_k}{\partial s_k} \neq \frac{\partial V_j}{\partial s_k}$ ,  $\frac{\partial V_k}{\partial s_j} \neq \frac{\partial V_j}{\partial s_j}$ .

**Definition 5** Let  $\Theta_{jk}(\mathbf{s}_{-jk})$  be the set of all interior  $\mathbf{s}_{jk}$ , such that  $(\mathbf{s}_{jk}, \mathbf{s}_{-jk})$  is strictly competitive for  $\{j, k\}$ . Profile  $\mathbf{s}'_{-jk}$  is **indistinguishable from**  $\mathbf{s}_{-jk}$  if for all  $\mathbf{s}_{jk} \in \Theta_{jk}(\mathbf{s}_{-jk}) \cup \Theta_{jk}(\mathbf{s}'_{-jk})$ ,  $\{j, k\} \subset \mathcal{I}(\mathbf{s}_{jk}, \mathbf{s}'_{-jk}) \cup \mathcal{I}(\mathbf{s}_{jk}, \mathbf{s}_{-jk})$ . By  $\Theta_{-jk}(\mathbf{s}_{-jk})$  we denote the set of all indistinguishable from  $\mathbf{s}_{-jk}$  profiles.

To see the intuition behind Definition 5, consider an interior  $\mathbf{s} = (\mathbf{s}_{jk}, \mathbf{s}_{-jk})$  that is strictly competitive for  $\{j, k\}$  and suppose that for some other  $\mathbf{s}'_{-jk}$  bidders  $\{-jk\}$  drop out at the same prices as if their signals were  $\mathbf{s}_{-jk}$ . Then, unless both  $j$  and  $k$  also have maximal values at  $(\mathbf{s}_{jk}, \mathbf{s}'_{-jk})$ , an efficient equilibrium does not exist. This is evident if neither  $j$  nor  $k$  has the highest value as they are the last two bidders in the auction, so suppose that only one of them, say  $k$ , does not have the highest value at  $(\mathbf{s}_{jk}, \mathbf{s}'_{-jk})$ . Because  $\mathbf{s} = (\mathbf{s}_{jk}, \mathbf{s}_{-jk})$  is strictly competitive for  $\{j, k\}$ , we can perturb  $\mathbf{s}_{jk}$  to obtain  $\mathbf{s}'_{jk}$ , so that: bidder  $k$  is the only bidder with the highest value at profile  $(\mathbf{s}'_{jk}, \mathbf{s}_{-jk})$ , while bidder  $j$  still has the highest value at profile  $(\mathbf{s}'_{jk}, \mathbf{s}'_{-jk})$ . These properties imply that to reach an efficient allocation at both of these profiles, neither bidder  $k$  nor  $j$  should be the first to drop out, but one of them necessarily does.

Note that existence of indistinguishable  $\mathbf{s}_{-jk}$  and  $\mathbf{s}'_{-jk}$  only implies that bidders  $\{-jk\}$  can exit identically at each profile without directly violating efficiency; it does not imply that they necessarily do so. Therefore, it is possible that only a subset  $\Theta$  of  $\Theta_{-jk}(\mathbf{s}_{-jk})$  cannot be inferred in equilibrium. At the same time, if  $\Theta$  contains two profiles in which two different bidders have two different signals, then, since each of the bidders bids identically for each of their signals,  $\Theta$  has to contain all possible combination pairs of such signals. If not, then as above, there is a combination of signals that results in the same bidding history

but has different bidders with the highest values. We call set  $\Theta$  a *product set* if it is a Cartesian product of individual sets of signals,  $\Theta = \times_{i \neq j, k} \Theta_i$ .

For the rest of the paper we are imposing the following assumption.

**Definition 6** *The signals' distribution  $f$  **preserves value properties** if, for any  $\{j, k\} \subset \mathcal{N}$ , any interior strictly competitive for  $\{j, k\}$  profile  $(\mathbf{s}_{jk}, \mathbf{s}_{-jk})$ , any product subset  $\Theta \subset \Theta_{-jk}(\mathbf{s}_{-jk})$ , and any  $\mathbf{s}'_{jk} \in \Theta_{jk}(\mathbf{s}_{-jk})$ , expected values  $W_j(\mathbf{s}'_{jk}) = E_{\mathbf{s}'_{-jk} \in \Theta} V_j(\mathbf{s}'_{jk}, \mathbf{s}'_{-jk})$  and  $W_k(\mathbf{s}'_{jk}) = E_{\mathbf{s}'_{-jk} \in \Theta} V_k(\mathbf{s}'_{jk}, \mathbf{s}'_{-jk})$  satisfy all the basic properties of the value functions.*

For generic value functions this property is trivially satisfied and imposes no restrictions on the signals' distribution. Indeed, it applies only if the set of indistinguishable from  $\mathbf{s}_{-jk}$  profiles,  $\Theta_{-jk}(\mathbf{s}_{-jk})$ , contains a non-singleton product subset  $\Theta$ .<sup>20</sup> Note also that, if the signals are independently drawn, both  $W_j$  and  $W_k$  keep the basic properties of  $V_j$  and  $V_k$  no matter over what sets the expectations are taken over.

### 4.3 Necessity Proposition

**Proposition 2 (Necessity)** *Suppose GSC is irreducibly violated at an interior profile  $\mathbf{s}$  for a subset  $\mathcal{A}$  and bidder  $k$ , and for some  $j \in \mathcal{A}$ ,  $\frac{\partial V_j(\mathbf{s})}{\partial s_k} \neq \frac{\partial V_k(\mathbf{s})}{\partial s_k}$ . Then no efficient equilibrium exists in the  $N$ -bidder English auction.*

The statement of the proposition differs from “if GSC is violated, then no efficient equilibrium exists” to accommodate the two (remaining) special cases in which an efficient equilibrium may actually exist. In the first case, as in Example 2, GSC is violated only at the boundary of the signals' domain. In the second case, as in Example 3, GSC is violated in the interior, but the nature of the violation is such that it is a “translation” of the violation on the boundary.<sup>21</sup> Our intuition suggests that in these two cases the efficiency properties of the English auction also depend on the properties of the values that are not maximal. While we do expect that a condition qualitatively similar to GSC ultimately determines whether an efficient equilibrium exists or not in these cases (plus some condition on signals' distribution in the second case),

<sup>20</sup> Note that the sets of indistinguishable profiles are degenerate (that is, singleton) for generic value functions. If, for instance, we slightly perturb all cross-partial derivatives by adding a (different) small constant to each of them, then, for almost all such perturbations all the sets of indistinguishable signals will be degenerate.

<sup>21</sup> If GSC is violated at interior  $\mathbf{s}$  but no interior  $\mathbf{s}'$  exists such that both conditions of the proposition hold, one can connect  $\mathbf{s}$  to a boundary profile  $\mathbf{s}^0$  with a continuous trajectory, such that along this trajectory GSC is violated for the same  $\mathcal{A}$  and  $k$ , and for all  $j \in \mathcal{A}$ ,  $\frac{\partial V_j(\mathbf{s})}{\partial s_k} = \frac{\partial V_k(\mathbf{s})}{\partial s_k}$ .

we choose to abstain from their analysis since it represents significant technical challenge without much additional insight.

The main implication of efficiency on bidding strategies can be described as: *after any history, and if active, the bidder who has strictly the highest value must not be the one to drop out first.* Suppose that at  $\mathbf{s}$  exactly one bidder, say bidder  $j$ , has the highest value among all the bidders,  $\#\mathcal{I}(\mathbf{s}) = 1$ . Then, in an efficient equilibrium bidder  $j$  must win the object, so at any intermediate history  $H(p)$  with  $\mathcal{M}$  being the set of active bidders,

$$\beta_j(s_j; H(p)) > \min_{i \in \mathcal{M}} \beta_i(s_i; H(p)). \quad (8)$$

When two or more bidders have the highest value at some  $\mathbf{s}$ , and the set of currently active bidders contains some bidder  $i \notin \mathcal{I}(\mathbf{s})$ , there must exist bidder  $j \in \mathcal{I}(\mathbf{s})$  for whom (8) holds. Note that it is not required that (8) holds for all bidders in the winners' circle.

On a high level, we prove Proposition 2 by establishing the following key facts.<sup>22</sup> We show that if GSC is satisfied at some interior  $\mathbf{s}$ , efficient equilibrium bidding strategies are locally increasing, and bidders drop out simultaneously whenever their signals are such that their values are equal. We also show that if GSC is violated at interior  $\mathbf{s}$  for bidders  $\mathcal{A}$  and  $k$ , while satisfied for any smaller subset  $\mathcal{B}$  ( $\#\mathcal{B} < \#\mathcal{A}$ ), then bidder  $k$  strategy is different. If  $\#\mathcal{A} = 1$ , that is if SC is violated, efficiency dictates that  $k$ 's strategy is decreasing at  $s_k$ , which cannot be in equilibrium. By waiting a little bit longer, bidder  $k$ , if wins, pays the price equal to her value evaluated at  $s'_k$  which is lower than the true  $s_k$ . Because her own signal has a positive effect on value, she gains since her actual value exceeds the price.

If  $\#\mathcal{A} > 1$ , perturb slightly the signals  $\mathbf{s}_{\mathcal{A}}$  in a way that the values of bidders  $\mathcal{A}$  increase uniformly. Bidder  $k$  becomes the only bidder with the highest value, and thus, she should not be the first to drop out. As before, bidders  $\mathcal{A}$  exit simultaneously whenever their signals are such that their values are equal given bidder  $k$ 's signal is (fixed at)  $s_k$ . But the same is true at any  $\mathbf{s}'$  sufficiently close to  $\mathbf{s}$ , such that GSC is violated at  $\mathbf{s}'$  for the same bidders and  $s'_k < s_k$ . Consider bidding at a profile  $(s'_k, \mathbf{s}_{-\mathcal{A}})$ : all bidders but  $k$  exit simultaneously, while  $k$  "waits" and wins. Since  $s'_k < s_k$  and SC is satisfied, her value is not the highest at  $(s'_k, \mathbf{s}_{-\mathcal{A}})$ , which contradicts efficiency.

Before proceeding with the proof we would like to describe in detail the notational convention we are going to use. It allows us to make strong statements about bidding functions in a simple and convenient form.

<sup>22</sup> For the purposes of this intuitive discussion, we assume that all GSC inequalities (2) are strict, ignore "for almost all" clauses as well as bidding by players outside the winners' circle.

Suppose that GSC is irreducibly violated at interior  $\mathbf{s}$  for subset  $\mathcal{A}$  and bidder  $k$ , and let  $\mathcal{C} = \mathcal{A} \cup \{k\} = \mathcal{I}(\mathbf{s})$ . If we fix  $\mathbf{s}_{-C}$  and slightly perturb  $\mathbf{s}_C$ , only bidders from  $\mathcal{C}$  can have the highest value. Thus, there exists an open neighborhood  $U_C^{\mathbf{s}}$  of  $\mathbf{s}_C$ , such that for any  $\mathbf{s}'_C \in U_C^{\mathbf{s}}$  and  $\mathbf{s}' = (\mathbf{s}'_C, \mathbf{s}_{-C})$ ,  $\mathcal{I}(\mathbf{s}') \subseteq \mathcal{I}(\mathbf{s})$ . For any such  $\mathbf{s}'$ , let  $p(\mathbf{s}')$  be the first price at which a bidder from  $\mathcal{C}$  drops out, so that  $H(p(\mathbf{s}'))$  is the sequence of drop out prices of bidders  $\mathcal{N} \setminus \mathcal{C}$  until the first bidder from  $\mathcal{C}$  drops out. Of course, it is not necessary that all the bidders  $\mathcal{N} \setminus \mathcal{C}$  exit first.

If it were the case that for any such  $\mathbf{s}'$  history  $H(p(\mathbf{s}'))$  is the same, then we could have fixed this history,  $H$ , and considered only parts of the strategies,  $\beta_j(s_j; H)$  for all  $j \in \mathcal{C}$ . Unfortunately, for different  $\mathbf{s}'$ ,  $H(p(\mathbf{s}'))$  may differ. For example, suppose that at  $\mathbf{s}$  bidder  $j$  is the first bidder from  $\mathcal{C}$  to drop out. This may no longer be so if we change slightly  $s_j$ . In such a case, depending on who is the next to drop out, the history either stays the same if it is a bidder from  $\mathcal{C}$  or lengthens if it is some other bidder not from  $\mathcal{C}$ . Even if bidder  $j$  is still the first from  $\mathcal{C}$  to drop out, the number of bidders from  $\mathcal{N} \setminus \mathcal{C}$  who drop out before  $j$  can decrease or increase.

To keep track of potentially different histories, and, therefore, different parts of the strategies, we propose the following. For any bidder  $j \in \mathcal{C}$  and signal  $s_j$  we define  $\hat{\beta}_j(s_j)$  as the price level at which she exits according to her equilibrium strategy if the other bidders from  $\mathcal{C}$  *remain active forever* (or do not exit before her) and all the bidders  $\mathcal{N} \setminus \mathcal{C}$  follow their equilibrium strategies (their signals are fixed). It is possible that  $\hat{\beta}_j(s_j) = \infty$  for some  $j$  and  $s_j$ . Note that the actual price at which the first bidder from  $\mathcal{C}$  drops out is equal to  $\min_{i \in \mathcal{C}} \hat{\beta}_i(s'_i)$ . In the proof we will be deriving results about *bidding functions*  $\hat{\beta}$ . For instance, if bidder  $j$  has strictly the highest value at  $\mathbf{s}'$ , he must not be the first to drop out, and so

$$\hat{\beta}_j(s'_j) > \min_{i \in \mathcal{C}} \hat{\beta}_i(s'_i).$$

To avoid excessive notation, we write simply  $\beta_j(s_j)$  in place of  $\hat{\beta}_j(s_j)$ . We are also omit the signals of the bidders  $\mathcal{N} \setminus \mathcal{C}$ , since these are going to be fixed. Thus,  $\mathbf{s}$  denotes the profile of signals of the bidders from  $\mathcal{C}$  only. We write  $\mathbf{s}_{\mathcal{N}}$  when referring to the full profile of signals.

**Proof of Proposition 2.** We proceed from the contrary, assuming that an efficient equilibrium exists. Fix an efficient equilibrium  $\beta$ . No restrictions on the bidding strategies are imposed, they need not be monotone and can be discontinuous everywhere.<sup>23</sup> We derive implications of GSC on bidding strategies and proceed by induction on the number of bidders in the winners' circle.

<sup>23</sup> If any of the bidders follows a mixed strategy, select an arbitrary strategy in the support of the mixed strategy.

The base of induction is the case  $\#\mathcal{I}(\mathbf{s}) = 2$ .

**Lemma 5** *Consider an interior strictly competitive for  $\{j, k\}$  signal profile  $\mathbf{s}$ . Then there is a neighborhood  $U^{\mathbf{s}} = U_j^{\mathbf{s}} \times U_k^{\mathbf{s}}$  such that: (1) for each  $s'_j \in U_j^{\mathbf{s}}$  there exists  $s'_k \in U_k^{\mathbf{s}}$  so that  $\{j, k\} = \mathcal{I}(s'_j, s'_k)$ ; (2) bidding functions of bidders  $j$  and  $k$  are monotone over  $U_j^{\mathbf{s}}$  and  $U_k^{\mathbf{s}}$ :  $\beta_j(s_j)$  is increasing (decreasing) whenever  $\frac{\partial V_j}{\partial s_j} > \frac{\partial V_k}{\partial s_j}$  ( $\frac{\partial V_j}{\partial s_j} < \frac{\partial V_k}{\partial s_j}$ ); (3) if  $\beta_j$  is continuous at  $s'_j \in U_j^{\mathbf{s}}$ , then  $\beta_k$  is continuous at  $s'_k \in U_k^{\mathbf{s}}$  such that  $\{j, k\} = \mathcal{I}(s'_j, s'_k)$ , and  $\beta_j(s'_j) = \beta_k(s'_k)$ .*

Note that we derive implications of SC at any strictly competitive signal profile, including those at which SC is strictly satisfied or violated for both bidders. The proof of Lemma 5 and the complete proof of the Proposition for the case  $\#\mathcal{A} = 1$  are presented in Appendix A.2.1. In particular, after establishing Lemma 5, we connect bidding functions of bidders  $j$  and  $k$  to their valuations. There are two cases: (1) for each  $s'_j \in U^{sj}$  there is only one signal  $s'_k$  of bidder  $k$  such that both  $j$  and  $k$  drop out simultaneously; and (2) there are more than one such signal for some  $s'_j \in U^{sj}$ . In the first case, we show that  $\beta_j(s'_j)$  must equal to the expected value of bidder  $j$  given  $s'_j$  and  $s'_k$ . In the second case,  $\beta_j(s'_j)$  must equal the expected value over all possible signals of bidder  $k$  that bid the same. Then, we argue that the second case cannot occur in an efficient equilibrium. Finally, if  $\mathbf{s}$  is an interior profile at which SC is violated for subset  $\{j\}$  and bidder  $k$  while  $\frac{\partial V_j}{\partial s_j} \neq \frac{\partial V_k}{\partial s_j}$ , then, since the strategy of bidder  $k$  is decreasing in  $s_k$ , bidder  $k$  has a profitable deviation.

The inductive step can be separated into the following 6 steps. Suppose that the irreducible subset  $\mathcal{A}$  contains  $n > 1$  bidders. For convenience, we relabel bidder  $k$  as bidder 1, and denote  $\mathcal{A} \cup \{1\}$  by  $\mathcal{A}^{+1}$ .

Step 1. Consider trajectory  $\mathbf{s}(t)$  that for each  $t$  solves

$$V_j(\mathbf{s}(t)) = V(\mathbf{s}) + t, \text{ for all } j \in \mathcal{A}^{+1}.$$

Such a trajectory exists and is unique, since it can be found as a solution to the differential equation

$$\frac{d\mathbf{s}}{dt} = (DV(\mathbf{s}))^{-1} \cdot \mathbf{1}. \quad (9)$$

By continuity of value functions and their first derivatives there exists an open neighborhood  $U_t^0$  of  $t = 0$  such that for all  $t \in U_t^0$ :  $\mathcal{A}^{+1} = \mathcal{I}(\mathbf{s}_{\mathcal{N}}(t))$ ; GSC is violated at  $\mathbf{s}_{\mathcal{N}}(t)$  for  $\mathcal{A}$  and 1; and  $\frac{\partial V_j(\mathbf{s}_{\mathcal{N}}(t))}{\partial s_1} \neq \frac{\partial V_1(\mathbf{s}_{\mathcal{N}}(t))}{\partial s_1}$  for some  $j \in \mathcal{A}$ .

Step 2. Consider  $\mathbf{s}' = \mathbf{s}(t)$  for an arbitrary  $t \in U_t^0$ . Lemma 9 in Appendix A.2.3 shows that for any  $j \in \mathcal{A}$ , there exist  $b_j(s'_j) \equiv \lim_{s_j \downarrow s'_j} \inf \beta_j(s_j)$ , and these limits are equal for the bidders from  $\mathcal{A}$ : for any  $j \in \mathcal{A}$ ,  $b_j(s'_j) \equiv b(\mathbf{s}') \equiv b(t) < \infty$ . In addition, for any  $j \in \mathcal{A}$  and  $s_j > s'_j$  sufficiently close to  $s'_j$ ,  $\beta_j(s_j) \geq b(t)$ ; and for bidder 1,  $\beta_1(s'_1) > b(t)$ .

Step 3. Corollary 3 in Appendix A.2.2 shows that for  $t' > t$  either: (i)  $s_1(t') < s_1(t)$  and, for all  $j \in \mathcal{A}$ ,  $s_j(t') > s_j(t)$ ; or (ii)  $s_1(t') > s_1(t)$  and, for all  $j \in \mathcal{A}$ ,  $s_j(t') < s_j(t)$ . This, together with Step 2, implies that  $b(t)$  is (weakly) monotonic in  $t$ . In Case (i) it is non-decreasing, in Case (ii) it is non-increasing.

Step 4. Corollary 4 in Appendix A.2.3 shows that if for some bidder  $j \in \mathcal{A}$ ,  $\beta_j(s_j(t)) \neq b(t)$ , then  $t$  has to be a discontinuity point for  $b(t)$ . Since  $b(t)$  is monotonic it has no more than a countable number of discontinuity points. Hence for almost all  $t \in U_t^0$ ,  $\beta_j(s_j(t)) = b(t)$  for every  $j \in \mathcal{A}$ . That is, when the signals of bidders from  $\mathcal{A}$  belong to trajectory  $\mathbf{s}(t)$ , bidders from  $\mathcal{A}$  almost always exit simultaneously.

Step 5. Consider two continuity points for  $b(t)$ ,  $t$  and  $t'$ , such that  $b(t') \geq b(t)$ . In Case (i),  $t' > t$ ; in Case (ii),  $t' < t$ . Then,  $s_1(t') < s_1(t)$ , and  $\beta_1(s_1(t')) > b(t') \geq b(t) = \beta_j(s_j(t))$  for all  $j \in \mathcal{A}$ .

Step 6. By construction, at  $t$ ,  $\mathcal{I}(s_1(t), \mathbf{s}_{\mathcal{A}}(t)) = \mathcal{A}^{+1}$ . We have

$$\frac{\partial V_1(s_1, \mathbf{s}_{\mathcal{A}}(t))}{\partial s_1} > \min_{j \in \mathcal{A}} \frac{\partial V_j(s_1, \mathbf{s}_{\mathcal{A}}(t))}{\partial s_1}. \quad (10)$$

Indeed, SC holds since  $\mathcal{A}$  is irreducible, and the equality is prevented by uniqueness of the solution to (9). Thus, if starting from  $\mathbf{s}(t)$  we slightly decrease the signal of bidder 1, she can no longer possess the highest value.

Then, we can find  $t'$  sufficiently close to  $t$  such that:  $t'$  is a continuity point for  $b(t)$ ,  $s_1(t') < s_1(t)$ , and all the bidders with the highest value at  $(s_1(t'), \mathbf{s}_{\mathcal{A}}(t))$  belong to  $\mathcal{A}$ . Then by results of Step 5, at the profile  $(s_1(t'), \mathbf{s}_{\mathcal{A}}(t))$ , bidders from  $\mathcal{A}$  drop out simultaneously at  $b(t)$  while bidder 1 for sure stays longer. Thus, efficiency is not achieved—a contradiction.  $\square$

## A Appendix

### A.1 Sufficiency

**Proof of Lemma 2.** Fix  $\mathbf{s}$ . First, we show that if GSC is satisfied, then  $\mathbf{u}_{\mathcal{A}}^{\mathcal{A}} \geq \mathbf{0}$  for all subsets  $\mathcal{A} \subset \mathcal{I}(\mathbf{s})$ . Then (5) follows from GSC verified on  $\mathbf{u}^{\mathcal{A}}$ .

Step 1. Suppose inequalities in (2) are strict for all  $\mathcal{A}$ . Then  $\mathbf{u}_{\mathcal{A}}^{\mathcal{A}} \gg \mathbf{0}$ , which we show by induction on  $\#\mathcal{A}$ . For  $\#\mathcal{A} = 1$ ,  $\mathbf{u}_{\mathcal{A}}^{\mathcal{A}} = \left(\frac{\partial V_{\mathcal{A}}}{\partial s_{\mathcal{A}}}\right)^{-1} > 0$ . Suppose that for  $n > 1$ , for all  $\mathcal{B} \subset \mathcal{I}(\mathbf{s})$  with  $\#\mathcal{B} \leq n - 1$ ,  $\mathbf{u}_{\mathcal{B}}^{\mathcal{B}} \gg \mathbf{0}$ . Suppose, on the contrary, there exists  $\mathcal{A} \subset \mathcal{I}(\mathbf{s})$  with  $\#\mathcal{A} = n$ , such that  $\mathbf{u}_{\mathcal{A}}^{\mathcal{A}} \not\gg \mathbf{0}$ . Partition  $\mathcal{A} = \mathcal{B} \sqcup \mathcal{C} \sqcup \mathcal{D}$ , so that  $\mathcal{B}$ ,  $\mathcal{C}$ , and  $\mathcal{D}$  are subsets for which the corresponding

components of  $\mathbf{u}_{\mathcal{A}}^A$  are negative, equal to zero, and positive. Thus,  $\mathcal{B} \cup \mathcal{C} \neq \emptyset$ . Note that also  $\mathcal{D} \neq \emptyset$ . (Otherwise, considering vector  $\mathbf{w} = -\mathbf{u}^A$ , we have  $\mathbf{w} \neq \mathbf{0}$ ,  $\mathbf{w}_{\mathcal{A}} \geq \mathbf{0}$ , and  $\mathbf{w}_{-\mathcal{A}} = \mathbf{0}$ . Therefore, there must exist  $j \in \mathcal{A}$  for whom  $\mu_j(\mathbf{w}) > 0$ , but  $\mu_j(\mathbf{w}) = -\mu_j(\mathbf{u}^A) = -1$  by definition of  $\mathbf{u}^A$ .) If  $\mathcal{B} = \emptyset$ , then  $\mathcal{C} \neq \emptyset$ , and for any  $j \in \mathcal{C}$ ,  $\mu_j(\mathbf{u}^D) = 1 = \max_{i \in \mathcal{D}} \{\mu_i(\mathbf{u}^D)\}$ , which contradicts the supposition that the inequalities in (2) are strict. Thus,  $\mathcal{B} \neq \emptyset$ . Consider  $\mathbf{u} \equiv (\mathbf{0}_{\mathcal{B}}, \mathbf{u}_{-\mathcal{B}}^A)$  and  $\mathbf{u}' \equiv (-\mathbf{u}_{\mathcal{B}}^A, \mathbf{0}_{-\mathcal{B}})$ . Note that,  $\mathbf{u}_{\mathcal{D}} \geq \mathbf{0}$ ,  $\mathbf{u}_{-\mathcal{D}} = \mathbf{0}$ ,  $\mathbf{u}'_{\mathcal{B}} \gg \mathbf{0}$ , and  $\mathbf{u}_{-\mathcal{B}} = \mathbf{0}$ . Consider bidder  $i \in \mathcal{B}$  with the maximal  $\mu_i(\mathbf{u}')$  and bidder  $j \in \mathcal{D}$  with the maximal  $\mu_j(\mathbf{u})$ . GSC for subset  $\mathcal{B}$  implies  $\mu_i(\mathbf{u}') > \mu_j(\mathbf{u}')$ , while GSC for subset  $\mathcal{D}$  implies  $\mu_i(\mathbf{u}) < \mu_j(\mathbf{u})$ . Since  $\mathbf{u}^A = \mathbf{u} - \mathbf{u}'$ , we have  $\mu_i(\mathbf{u}^A) < \mu_j(\mathbf{u}^A)$ , which is a contradiction as  $\mu_i(\mathbf{u}^A) = \mu_j(\mathbf{u}^A) = 1$  by definition of  $\mathbf{u}^A$ . Therefore,  $\mathbf{u}_{\mathcal{A}}^A \gg \mathbf{0}$ .

Step 2. If some of the inequalities in (2) are weak, we can slightly perturb  $DV_{\mathcal{I}}(\mathbf{s})$  by adding  $\varepsilon > 0$  to every diagonal element,  $DV'_{\mathcal{I}}(\mathbf{s}) = DV_{\mathcal{I}}(\mathbf{s}) + \varepsilon I_{\#\mathcal{I}}$ , and all inequalities in (2) become strict. (For any  $\mathcal{A} \subset \mathcal{I}(\mathbf{s})$  and  $\mathbf{u}$  with  $\mathbf{u}_{-\mathcal{A}} = \mathbf{0}$  we have  $\mu'_j(\mathbf{u}) = \varepsilon \frac{\partial V_j}{\partial s_j} + \mu_j(\mathbf{u}) > \mu_j(\mathbf{u})$  for all  $j \in \mathcal{A}$ , while  $\mu'_i(\mathbf{u}) = \mu_i(\mathbf{u})$  for all  $i \notin \mathcal{A}$ .) From Step 1 we have  $\mathbf{u}_{\mathcal{A}}^A(\varepsilon) \gg \mathbf{0}$ . By continuity,  $\mathbf{u}_{\mathcal{A}}^A = \lim_{\varepsilon \rightarrow 0} \mathbf{u}_{\mathcal{A}}^A(\varepsilon) \geq \mathbf{0}$  for all subsets  $\mathcal{A} \subset \mathcal{I}(\mathbf{s})$ .

Second, also by induction on  $\#\mathcal{A}$ , we show the “only if” part of the Lemma. For  $\#\mathcal{A} = 1$  the result is obvious. Suppose that GSC is satisfied at  $\mathbf{s}$  for all  $\mathcal{B} \subset \mathcal{I}(\mathbf{s})$  with  $\#\mathcal{B} < n$ , and that GSC is violated at  $\mathbf{s}$  for some  $\mathcal{A} \subset \mathcal{I}(\mathbf{s})$  with  $\#\mathcal{A} = n > 1$ . That is, there exists a vector  $\mathbf{u}$  with  $\mathbf{u}_{\mathcal{A}} \gg \mathbf{0}$  and  $\mathbf{u}_{-\mathcal{A}} = \mathbf{0}$ , such that for some  $k \in \mathcal{I}(\mathbf{s}) \setminus \mathcal{A}$ ,  $\mu_k(\mathbf{u}) > \max_{j \in \mathcal{A}} \mu_j(\mathbf{u})$ . Clearly,  $\mathbf{u} \neq \mathbf{u}^A$ .

Consider  $\mathcal{B} \subset \mathcal{A}$ , such that  $i \in \mathcal{B} \Leftrightarrow \mu_i(\mathbf{u}) = \max_{j \in \mathcal{A}} \mu_j(\mathbf{u})$ . Since  $\mathbf{u} \neq \mathbf{u}^A$ ,  $\mathcal{B} \neq \mathcal{A}$ , and by the above argument,  $\mathbf{u}_{\mathcal{B}}^B \geq \mathbf{0}$ . Consider vector  $\mathbf{w}_1(t) = \mathbf{u} - t\mathbf{u}^B$  (note that since  $\mathcal{B} \subset \mathcal{A}$ ,  $\mathbf{u}_{\mathcal{B}} \gg \mathbf{0}$ ). At  $t = 0$ , for any  $j \in \mathcal{A} \setminus \mathcal{B}$  and  $i \in \mathcal{B}$ , we have  $\mu_j(\mathbf{w}_1(0)) < \mu_i(\mathbf{w}_1(0)) < \mu_k(\mathbf{w}_1(0))$ . Once we start increasing  $t$ , thus decreasing the signals of all the bidders from  $\mathcal{B}$  only, all  $\mu_i(\mathbf{w}_1(t))$  decrease uniformly at rate  $t$ , while for any bidder  $l \in \mathcal{I}(\mathbf{s}) \setminus \mathcal{B}$  (including  $k$ ) her  $\mu_l(\mathbf{w}_1(t))$  decreases at most at the same rate, because GSC is satisfied for  $\mathcal{B}$ . Let  $t_1$  be the minimal  $t > 0$  such that: either  $\mu_j(\mathbf{w}_1(t)) = \mu_i(\mathbf{w}_1(t))$  for some  $j \in \mathcal{A} \setminus \mathcal{B}$  or  $w_{1i}(t) = 0$  for some  $i \in \mathcal{B}$ . In the latter case, stop. Otherwise, consider  $\mathcal{C}$ —a subset that includes  $\mathcal{B}$  and all  $j \in \mathcal{A} \setminus \mathcal{B}$  for whom  $\mu_j(\mathbf{w}_1(t_1)) = \mu_i(\mathbf{w}_1(t_1))$ . Define  $\mathbf{w}_2(t) = \mathbf{w}_1(t_1) - t\mathbf{u}^{\mathcal{C}}$ . Find the minimal  $t_2 > 0$  such that: either  $\mu_j(\mathbf{w}_2(t_2)) = \mu_i(\mathbf{w}_2(t_2))$  for some  $j \in \mathcal{A} \setminus \mathcal{C}$  and all  $i \in \mathcal{C}$ , or  $w_{2i}(t_2) = 0$  for some  $i \in \mathcal{C}$ . Again, in the latter case stop, otherwise consider  $\mathcal{D} \supset \mathcal{C}$ , and so on. Repeat this procedure until for some  $i \in \mathcal{A}$ ,  $w_{mi}(t_m) = 0$ , which will take at most  $\#\mathcal{A}$  repetitions. Note that  $\mu_k(\mathbf{w}_1(t))$  has always decreased at a rate no higher than the corresponding rate for bidders from  $\mathcal{B}, \mathcal{C}, \mathcal{D}, \dots$ . Thus, at any stage  $l \leq m$  of the procedure,  $\mu_k(\mathbf{w}_l(t)) > \max_{j \in \mathcal{A}} \mu_j(\mathbf{w}_l(t))$ . In particular,

$$\mu_k(\mathbf{w}_m(t_m)) > \max_{j \in \mathcal{A}} \mu_j(\mathbf{w}_m(t_m)). \quad (\text{A.1})$$

If for all  $j \in \mathcal{A}$ ,  $w_{mj}(t_m) = 0$ , then, by construction,  $\mathbf{w}_m(t_m) = \mathbf{0}$ , which makes (A.1) impossible. Else, define  $\mathcal{A}'$  to be the set of bidders  $j \in \mathcal{A}$  with  $w_{mj}(t_m) > 0$ . Since  $\#\mathcal{A}' < n$ , GSC is violated for the set  $\mathcal{A}'$ , vector  $\mathbf{w}_m(t_m)$ , and bidder  $k \in \mathcal{I}(s) \setminus \mathcal{A}$ , which contradicts the induction presumption.  $\square$

**Proof of Lemma 3.** First, we construct inferences  $\boldsymbol{\sigma}(p, H(p))$ , such that each  $\sigma_j(\cdot, H(p))$  is continuous in  $p$  for any  $H(p)$ , and any active bidder  $j$  with  $\sigma_j(p, H(p)) < 1$  has the highest value at  $\boldsymbol{\sigma}(p)$ . Define  $\mathcal{A}(\boldsymbol{\sigma}(p, H(p)))$  to consist of all  $j$ , who are active at  $p$  and for whom  $\sigma_j(p, H(p)) < 1$ .

Suppose that at some  $p^0$  with  $H(p^0) = \bar{H}(p^0)$  there exists a profile  $\boldsymbol{\sigma}^0(p^0)$  satisfying (3), and  $\mathcal{A} = \mathcal{A}(\boldsymbol{\sigma}^0(p^0)) \subset \mathcal{I}(\boldsymbol{\sigma}^0(p^0))$ . Fix  $\boldsymbol{\sigma}_{-\mathcal{A}}(p) = \boldsymbol{\sigma}_{-\mathcal{A}}^0(p^0)$  for  $p \geq p^0$ . First, we find a solution  $\boldsymbol{\sigma}_{\mathcal{A}}(p)$  to the system

$$\mathbf{V}_{\mathcal{A}}(\boldsymbol{\sigma}_{\mathcal{A}}(p), \boldsymbol{\sigma}_{-\mathcal{A}}^0(p^0)) = p \mathbf{1}_{\mathcal{A}}, \quad (\text{A.2})$$

which is equivalent to solving the system of differential equations

$$\frac{d\boldsymbol{\sigma}_{\mathcal{A}}}{dp} = (DV_{\mathcal{A}})^{-1} \mathbf{1}_{\mathcal{A}}. \quad (\text{A.3})$$

By the Cauchy-Peano theorem, given  $\boldsymbol{\sigma}_{\mathcal{A}}(p^0) = \boldsymbol{\sigma}_{\mathcal{A}}^0(p^0)$ , there exists a unique continuous solution  $\boldsymbol{\sigma}_{\mathcal{A}}(p)$  to (A.3) on  $p \in [p^0, p^*]$ , for some  $p^* > p^0$ .

Suppose GSC is satisfied. As long as  $\mathcal{A} \subset \mathcal{I}(\boldsymbol{\sigma}(p))$ ,  $\frac{d\boldsymbol{\sigma}_{\mathcal{A}}}{dp} = \mathbf{u}_{\mathcal{A}}^{\mathcal{A}} \geq \mathbf{0}$  (from the proof of Lemma 2) and  $\frac{\partial V_i}{\partial \boldsymbol{\sigma}_{\mathcal{A}}} \frac{d\boldsymbol{\sigma}_{\mathcal{A}}}{dp} \leq 1$  for any  $i \in \mathcal{I}(\boldsymbol{\sigma}(p)) \setminus \mathcal{A}$  by (5). As a result,  $\boldsymbol{\sigma}(p)$  extends to all  $p \leq p_{\mathcal{A}}^*$ , where  $p_{\mathcal{A}}^*$  is the lowest price at which  $\sigma_j(p_{\mathcal{A}}^*) = 1$  for some  $j \in \mathcal{A}$ . To extend  $\boldsymbol{\sigma}(p, H(p))$  beyond  $p_{\mathcal{A}}^*$ , a new system (A.2) is solved for  $\mathcal{A}' = \mathcal{A}(\boldsymbol{\sigma}(p_{\mathcal{A}}^*, H(p_{\mathcal{A}}^*))) \subsetneq \mathcal{A}$  with the initial condition  $\boldsymbol{\sigma}_{\mathcal{A}'}^0(p_{\mathcal{A}}^*) = \boldsymbol{\sigma}_{\mathcal{A}}(p_{\mathcal{A}}^*)$ . This is repeated until there is no  $j$  with  $\sigma_j(p) < 1$ , after which  $\boldsymbol{\sigma}(p)$  is fixed.

To provide  $\boldsymbol{\sigma}(p, H(p))$  for all prices and histories we need to specify for each  $H(p)$  the initial  $\boldsymbol{\sigma}^0(p^0)$  for  $p^0 = \max_{p_j \in H(p)} p_j$ . Set  $\boldsymbol{\sigma}^0(0, \emptyset) = \mathbf{0}$  and compute  $\boldsymbol{\sigma}(p)$  for all  $p > 0$ , starting with  $\mathcal{A} = \mathcal{N}$ . At  $p^*$  such that  $H(p^*) \neq \bar{H}(p^*)$ , define  $\boldsymbol{\sigma}^0(p^*, \bar{H}(p^*)) = \boldsymbol{\sigma}(p^*, H(p^*))$  (which maintains continuity of inferences), and compute  $\boldsymbol{\sigma}(p, \bar{H}(p^*))$  for all  $p > p^*$ .  $\square$

## A.2 Necessity

**Proof of Lemma 1.** Fix  $\varepsilon > 0$  so that all profiles within  $\varepsilon$  of  $\mathbf{s}$  are interior. Among these profiles, select  $\mathbf{s}^*$ , at which a violation of GSC involves the smallest number of bidders, and let  $\mathcal{A}^*$  and  $k^*$  be these bidders. If  $\mathcal{A}^* \cup k^* = \mathcal{I}(\mathbf{s}^*)$ , then set  $\mathbf{s}' = \mathbf{s}^*$  and choose any  $U^{\mathbf{s}'}$  fully inside  $\varepsilon$ -ball of  $\mathbf{s}$ . If  $\mathcal{A}^* \cup k^* \neq \mathcal{I}(\mathbf{s}^*)$ , let  $\mathcal{C} = \mathcal{I}(\mathbf{s}^*) \setminus (\mathcal{A}^* \cup k^*)$ . While keeping the values of bidders  $\mathcal{A}^* \cup k^*$  and

signals of bidders not in  $\mathcal{A}^* \cup k^* \cup \mathcal{C}$  fixed, reduce the values of all the bidders from  $\mathcal{C}$  uniformly to obtain profile  $\mathbf{s}'$  such that  $|\mathbf{s}' - \mathbf{s}| < \varepsilon$  and GSC is violated at  $\mathbf{s}'$  for  $\mathcal{A}^*$  and  $k^*$ . Such a reduction is possible because value functions are  $C^1$  and regular—indeed, there is a unique way to adjust the signals of bidders  $\mathcal{A}^* \cup k^* \cup \mathcal{C}$  corresponding to the desired adjustment in values.  $\square$

### A.2.1 Proof of Proposition 2 for $\#\mathcal{A} = 1$

**Proof of Lemma 5.** Consider trajectory  $\mathbf{s}(\tau) = (s_j(\tau), s_k(\tau))$  such that  $V_j(\mathbf{s}(\tau)) = V_k(\mathbf{s}(\tau)) = v + \tau$ , where  $v = V_j(\mathbf{s})$ . Since value functions are  $C^1$ , such a trajectory is well defined (exists and is unique) in some neighborhood of  $\mathbf{s}$  (equivalently of  $\tau = 0$ ) and, since the single-crossing inequalities (2) for  $j$  and  $k$  are strict, it is not “flat” at  $\tau = 0$ . Thus, one can find a sufficiently small neighborhood  $(\tau_-, \tau_+)$  of  $\tau = 0$  and corresponding  $U^{\mathbf{s}} = s_j(\tau_-, \tau_+) \times s_k(\tau_-, \tau_+)$ ,<sup>24</sup> so that: (1)  $s_j(\tau)$  and  $s_k(\tau)$  are strictly monotone on  $(\tau_-, \tau_+)$ ; (2)  $U^{\mathbf{s}}$  is interior, that is  $U^{\mathbf{s}} \in (0, 1) \times (0, 1)$ ; (3) the single-crossing inequalities are of the same sign at all  $\mathbf{s}'$  in  $U^{\mathbf{s}}$ ; and (4) for all  $(s'_j, s'_k) \in U^{\mathbf{s}}$ ,  $\mathcal{I}(s'_j, s'_k) = \{j, k\} \Leftrightarrow (s'_j, s'_k) = \mathbf{s}(\tau)$  for some  $\tau \in (\tau_-, \tau_+)$ . Thus,  $U^{\mathbf{s}}$  is being split by  $\mathbf{s}(\tau)$  in two halves, in one half bidder  $j$  has the highest value, in the other half bidder  $k$  has the highest value.

Pick any  $\tau', \tau'' \in (\tau_-, \tau_+)$  so that  $s_j(\tau') < s_j(\tau'')$  and any  $\tau^*$  between  $\tau'$  and  $\tau''$ . Suppose that  $\frac{\partial V_j}{\partial s_j} < \frac{\partial V_k}{\partial s_j}$ , then at  $(s_j(\tau'), s_k(\tau^*))$  bidder  $j$  has the highest value, while at  $(s_j(\tau''), s_k(\tau^*))$  bidder  $k$  does. Thus,

$$\beta_j(s'_j) > \beta_k(s'_k) > \beta_j(s''_j). \quad (\text{A.4})$$

That is,  $j$ 's bidding function is decreasing. Similarly, if  $\frac{\partial V_j}{\partial s_j} > \frac{\partial V_k}{\partial s_j}$ ,  $\beta_j$  is increasing, and the same holds for bidder  $k$ . A monotone function is continuous almost anywhere. Thus, from (A.4), if  $\beta_j$  is continuous at  $s_j(\tau)$ , then  $\beta_j(s_j(\tau)) = \beta_k(s_k(\tau))$ , else  $\beta_k$  is discontinuous at  $s_k(\tau)$  as well.  $\square$

**Corollary 1** Consider an interior strictly competitive for  $\{j, k\}$  profile  $\mathbf{s}$  and the neighborhood  $U^{\mathbf{s}}$  as in Lemma 5. Then, if  $\#\mathcal{N} > 2$ , for any  $\mathbf{s}'_{jk} \in U^{\mathbf{s}}$ , all the other bidders drop out before both  $j$  and  $k$  (recall that  $\mathbf{s}_{-jk}$  is fixed),

$$\max_{i \neq j, k} \beta_i(s_i) \leq \inf_{(s'_j, s'_k) \in U^{\mathbf{s}}} \{\beta_j(s'_j), \beta_k(s'_k)\}.$$

**Proof.** If there exists  $i \neq j, k$  with  $\beta_i(s_i) > \inf_{(s'_j, s'_k) \in U^{\mathbf{s}}} \{\beta_j(s'_j), \beta_k(s'_k)\}$ , then there exists  $\tau$ , such that  $\beta_j(s_j(\tau)) = \beta_k(s_k(\tau)) < \beta_i(s_i)$ —bidders  $j$  and  $k$  exit simultaneously while bidder  $i$  remains active, which contradicts efficiency.  $\square$

<sup>24</sup>For any function  $f : X \rightarrow Y$  and subsets  $S \subset X$  and  $T \subset Y$ , we define  $f(S) = \{f(x) : x \in S\}$  and  $f^{-1}(T) = \{x : f(x) \in T\}$ .

**Proof of Proposition 2 for  $\#\mathcal{A} = 1$ .** Letting  $\mathcal{A} = \{j\}$  and relying on Lemma 5, there are two cases: (1) for each  $s'_j \in U_j^s$  there is a single  $s'_k$ , such that both  $j$  and  $k$  drop out simultaneously; and (2) for some  $s'_j \in U_j^s$  there are multiple such signals. Let  $\Theta_{-jk}$  be the set of profiles  $\mathbf{s}'_{-jk}$  for which the history is the same as for  $\mathbf{s}_{-jk}$ . Clearly,  $\Theta_{-jk}$  must be a product subset of  $\Theta_{-jk}(\mathbf{s}_{-jk})$  (see the argument following Definition 5). By Corollary 1, at any  $\mathbf{s}_{-jk} \in \Theta_{-jk}$  bidders  $\{-jk\}$  exit prior to both  $j$  with  $s_j \in U_j^s$  and  $k$  with  $s_k \in U_k^s$ .

**Case 1.** In general,  $\beta_j^{-1}(\beta_j(U_j^s))$  need not be equal to  $U_j^s$  as bids at other signals may also lie within  $\beta_j(U_j^s)$ . Let  $CU_j^s \subset U_j^s$  and  $CU_k^s \subset U_k^s$  be sets of signals over which, respectively,  $\beta_j$  and  $\beta_k$  are continuous.

**Lemma 6** *Consider an interior strictly competitive signal profile  $\mathbf{s}$ . Suppose that there is a neighborhood  $U^s$  such that  $U_k^s \supset \beta_k^{-1}(\beta_j(CU_j^s))$  and conditions of Lemma 5 are satisfied. Let  $W_j(s_j, s_k) = E_{\mathbf{s}'_{-jk} \in \Theta_{-jk}(s_j, s_k)} V_j(s_j, s_k, \mathbf{s}'_{-jk})$ .<sup>25</sup> Then, for all  $s'_j \in CU_j^s$  and  $s'_k$  such that  $\beta_j(s'_j) = \beta_k(s'_k)$ ,  $\beta_j(s'_j) = W_j(s'_j, s'_k)$ . Also,  $CU_j^s = U_j^s$  and  $CU_k^s = U_k^s$ .*

**Proof.** Consider  $(s'_j, s'_k) \in U^s$  such that  $\beta_j$  and  $\beta_k$  are continuous, respectively, at  $s'_j$  and  $s'_k$  (thus  $\beta_j(s'_j) = \beta_k(s'_k)$ ). Suppose  $\beta_j(s'_j) < W_j(s'_j, s'_k)$  and consider the following deviation of bidder  $j$  with  $s'_j$ : increase the bid by  $\varepsilon > 0$ . By doing so,  $j$  wins and pays the same at all realizations she was winning before. In addition, she also wins when  $s_k$  is such that  $\beta_k(s_k) \in (\beta_j(s'_j), \beta_j(s'_j) + \varepsilon)$ . If  $\varepsilon$  is sufficiently small, such  $s_k$  must be close to  $s'_k$  since  $\beta_k$  is monotone and  $U_k^s = \beta_k^{-1}(\beta_k(CU_k^s))$  (recall that  $\beta_k(CU_k^s) = \beta_j(CU_j^s)$ ). Bidder  $j$  obtains value  $W_j(s'_j, s_k)$  that is close to  $W_j(s'_j, s'_k)$  and pays  $\beta_k(s_k)$ . When  $\varepsilon$  is sufficiently small,  $\beta_k(s_k) < \beta_j(s'_j) + \varepsilon < W_j(s'_j, s_k)$ . Since for any  $\varepsilon > 0$  the probability of  $s_k$  with  $\beta_k^{-1}(\beta_j(s'_j), \beta_j(s'_j) + \varepsilon)$  is positive, the deviation is profitable. Similarly, if  $\beta_j(s'_j) > W_j(s'_j, s'_k)$ , then it is profitable for bidder  $j$  to exit earlier. Continuity of  $\beta_j$  and  $\beta_k$  on, respectively, the whole  $U_j^s$  and  $U_k^s$  follows from their monotonicity and continuity of  $W_j$  and  $W_k$ .  $\square$

**Corollary 2** *No efficient equilibrium exists under conditions of Lemma 6.*

**Proof.** By Lemmas 5 and 6,  $\beta_j(s_j(\tau)) = \beta_k(s_k(\tau)) = W_j(\mathbf{s}(\tau)) = W_k(\mathbf{s}(\tau))$  for all  $\tau \in (\tau_-, \tau_+)$ . Because the signals' distribution preserves value properties,  $s_j(\tau)$  and  $s_k(\tau)$  cannot be both decreasing, since  $W_j$  and  $W_k$  are increasing, or both increasing, since decreasing  $\beta_i(s_j(\tau))$  is incompatible with increasing  $W_j(\mathbf{s}(\tau))$ . If  $s_j(\tau)$  is decreasing, then  $s_k(\tau)$  is increasing,  $\frac{\partial V_k(\mathbf{s}(\tau))}{\partial s_k} > \frac{\partial V_j(\mathbf{s}(\tau))}{\partial s_k}$ , and  $\beta_k$  is increasing. Bidder  $j$  with  $s_j(\tau)$  can improve her payoff by staying longer: by bidding  $\beta_j(s_j(t))$  for some  $t \in (\tau, \tau_+)$ , she also wins against  $k$  with  $s_k(t')$  for  $t' \in (\tau, t)$  and pays  $W_j(s_j(t'), s_k(t')) < W_j(s_j(\tau), s_k(t'))$ . (The last inequality is true because  $f$  preserves value properties. Note that this is so

<sup>25</sup> The expectation is based on posterior probabilities of  $\mathbf{s}'_{-jk}$  given the history.

even if some of the bidders  $\{-ij\}$  follow mixed strategies, since the mixing by all types and all players is done independently.) If  $s_j(\tau)$  is increasing, then  $s_k(\tau)$  is decreasing and  $\frac{\partial V_k(\mathbf{s}(\tau))}{\partial s_k} > \frac{\partial V_j(\mathbf{s}(\tau))}{\partial s_k}$ . Again, bidder  $j$  with  $s_j(\tau)$  improves by staying longer: by bidding  $\beta_j(s_j(t))$  for  $t \in (\tau_-, \tau)$ , she also wins against all  $k$  with  $s_k(t')$  for  $t' \in (t, \tau)$  and pays  $W_j(s_j(t'), s_k(t')) < W_j(s_j(\tau), s_k(t'))$ .  $\square$

**Case 2.** Suppose there exist  $s_j(\tau)$  and multiple  $s'_k$  such that  $\beta_k(s'_k) = \beta_j(s_j(\tau))$ . For each  $\tau \in (\tau_-, \tau_+)$ , define  $\Theta_k(\tau) = \{s'_k : \beta_j(s_j(\tau)) = \beta_k(s'_k)\}$ . Note that, for all  $s'_k \in \Theta_k(\tau)$ ,  $\{j, k\} \in \mathcal{I}(s_j(\tau), s'_k)$ . (Indeed, as  $j$  and  $k$  are the last bidders to drop out one of them must have the highest value. If, for instance,  $V_k(s_j(\tau), s'_k) < V_j(s_j(\tau), s'_k)$ , then one can select  $s'_j$  sufficiently close to  $s_j(\tau)$ , so that  $\beta_j(s'_j) < \beta_j(s_j(\tau)) = \beta_k(s'_k)$  but still  $V_k(s'_j, s'_k) < V_j(s'_j, s'_k)$ , which would contradict efficiency.) In general, for some  $\tau$ ,  $s'_j = s_j(\tau)$ , and  $s'_k \in \Theta_k(\tau)$ , profile  $(s'_j, s'_k)$  may not be strictly competitive. There are three possibilities: (1)  $\frac{\partial V_k(s'_j, s'_k)}{\partial s_k} = \frac{\partial V_j(s'_j, s'_k)}{\partial s_k}$ ; (2)  $\frac{\partial V_k(s'_j, s'_k)}{\partial s_j} = \frac{\partial V_j(s'_j, s'_k)}{\partial s_j}$ ; or (3)  $\mathcal{I}(\mathbf{s}') \setminus \{j, k\} \neq \emptyset$ . We show that all of these special cases can be avoided by a careful choice of  $\tau$  and, possibly, a slight perturbation of the original  $\mathbf{s}$ , so that all  $(s'_j, s'_k)$  (for this  $\tau$ ) are strictly competitive for  $\mathbf{s}_{-jk}$ .

Let  $SS_j \subset U_j^{\mathbf{s}}$  consist of all  $s_j(\tau)$  for which there exists  $s'_k \in \Theta_k(\tau)$  such that  $\frac{\partial V_j(\mathbf{s}')}{\partial s_k} = \frac{\partial V_k(\mathbf{s}')}{\partial s_k}$ . This set is finite. If not, select  $(s_j^n, s_k^n) \rightarrow_{n \rightarrow \infty} (s_j^*, s_k^*)$  such that  $\forall n, m \in \mathbb{N}$ ,  $s_j^n = s_j(\tau^n) \in SS_j$ ,  $s_j^n \neq s_j^m$  for  $n \neq m$ , and  $s_k^n \in \Theta_k(\tau^n)$ . Since value functions are  $C^1$ ,  $\frac{\partial V_j(s_j^n, s_k^n)}{\partial s_k} = \frac{\partial V_k(s_j^n, s_k^n)}{\partial s_k}$ ,  $V_j(s_j^n, s_k^n) = V_k(s_j^n, s_k^n) \equiv V^*$ ,  $\{j, k\} \in \mathcal{I}(s_j^n, s_k^n)$ , and, by regularity,  $\frac{\partial V_j(s_j^n, s_k^n)}{\partial s_j} \neq \frac{\partial V_k(s_j^n, s_k^n)}{\partial s_j}$ . Consider trajectory  $\mathbf{s}'(t)$  defined as  $V_j(\mathbf{s}'(t)) = V_k(\mathbf{s}'(t)) = V^* + t$ . Such trajectory is unique in the neighborhood of  $(s_j^*, s_k^*) = \mathbf{s}'(0)$  (by regularity), thus, there exists  $M$  that for all  $n > M$ ,  $(s_j^n, s_k^n)$  belongs to the trajectory. (If  $(s_j^*, s_k^*)$  is on the boundary, the trajectory is guaranteed to exist only for  $t > 0$  or  $t < 0$  by the very existence of the above converging sequence.) Without loss of generality suppose that all  $(s_j^n, s_k^n)$  are on one side of this trajectory (otherwise select a converging subsequence that is on one side), and that this side corresponds to  $t > 0$  (the argument for  $t < 0$  is similar). Then, there exists  $\varepsilon > 0$  such that for all  $t \in (0, \varepsilon)$  either  $\frac{\partial V_j(\mathbf{s}'(t))}{\partial s_k} = \frac{\partial V_k(\mathbf{s}'(t))}{\partial s_k}$  or  $\frac{\partial V_j(\mathbf{s}'(t))}{\partial s_k} \neq \frac{\partial V_k(\mathbf{s}'(t))}{\partial s_k}$ . In the former case,  $s'_j(t) = s_j^*$  and for all  $n, m > M_1(\varepsilon)$  we have  $s_j^n = s_j^m$ , which is a contradiction; in the latter case, for  $n > M_2(\varepsilon)$ ,  $(s_j^n, s_k^n)$  is not a ‘‘special’’ profile. Since  $SS_j$  and  $SS_k$  (similarly defined) are finite, one can select  $\tilde{\mathbf{s}}$  on trajectory  $\mathbf{s}(\tau)$  and  $U^{\tilde{\mathbf{s}}} \subset U^{\mathbf{s}}$ , such that  $U_j^{\tilde{\mathbf{s}}} \subset CU_j^{\mathbf{s}} \setminus SS_j$  and  $U_k^{\tilde{\mathbf{s}}} \subset CU_k^{\mathbf{s}} \setminus SS_k$ . Without loss of generality one can reset  $\mathbf{s} = \tilde{\mathbf{s}}$ , and  $U^{\mathbf{s}} = U^{\tilde{\mathbf{s}}}$  (by construction,  $\tilde{\mathbf{s}}$  satisfies conditions of Proposition 2). Thus, both  $\beta_j$  and  $\beta_k$  are continuous over, respectively,  $U_j^{\mathbf{s}}$  and  $U_k^{\mathbf{s}}$ , and no ‘‘special’’ signals belong to  $U_i^{\mathbf{s}}$  and  $U_k^{\mathbf{s}}$ .

Next, note that for each  $s_j(\tau)$ ,  $\Theta_k(\tau)$  is finite. (If  $\Theta_k(\tau)$  is infinite, one can consider a sequence of  $s_k$  from  $\Theta_k(\tau)$  converging to  $s_k^*$ . In the limit,  $\frac{\partial V_k(s_j(\tau), s_k^*)}{\partial s_k} =$

$\frac{\partial V_j(s_j(\tau), s_k^*)}{\partial s_k}$ , which contradicts  $s_j(\tau) \notin SS_j$ .) For an arbitrary  $s'_j = s_j(\tau)$  and  $s'_k \in \Theta_k(\tau)$  define trajectory  $\mathbf{s}'(t) = (s'_j(t), s'_k(t))$ , such that  $V_j(\mathbf{s}'(t)) = V_k(\mathbf{s}'(t)) = v + t$ , where  $v = V_j(s'_j, s'_k)$ . It is well defined since  $j, k \in \mathcal{I}(s'_j, s'_k)$ . Then, since value functions are  $C^1$ , there exists  $\bar{t} > 0$ , such that for every  $t \in (0, \bar{t})$  either (1)  $\mathcal{I}(\mathbf{s}'(t)) = \mathcal{B}$  and  $j, k \in B$  or (2)  $j, k \notin \mathcal{I}(\mathbf{s}'(t))$ . In the latter case, for a sufficiently small  $0 < t < \bar{t}$ , for  $s_j(\hat{\tau}) = s_j(t)$  there will be no  $\hat{s}_k \in \Theta(\hat{\tau})$  close to  $s'_k$ . In the former case, if  $\mathcal{D} = \mathcal{B} \setminus \{j, k\} \neq \emptyset$ , then starting with  $(s'_j, s'_k, \mathbf{s}_{-jk})$  one can slightly reduce values of bidders  $\mathcal{D}$  uniformly, while keeping the values of  $\{j, k\}$  and signals of all other bidders fixed, to obtain  $\hat{\mathbf{s}}'$  so that  $\mathcal{I}(\hat{\mathbf{s}}') = \{j, k\}$ . Define  $\tilde{s}_{-k} = \hat{s}'_{-k}$  and  $\tilde{s}_k = s_k(\tau')$ , where  $\tau'$  solves  $s_j(\tau') = \hat{s}'_j$ . If the reduction is sufficiently small, all of the original and derived properties (of valuations and bidding functions) are the same at  $\tilde{\mathbf{s}}$  and  $\mathbf{s}(\tau)$ , while  $\hat{\mathbf{s}}'$  is strictly competitive for  $\{j, k\}$ . Finally, if  $\frac{\partial V_k(\mathbf{s}'(t))}{\partial s_j} = \frac{\partial V_j(\mathbf{s}'(t))}{\partial s_j}$  at  $t = 0$ , then there exists  $t^* > 0$ , that either (1) this is also true for all  $t \in (0, t^*)$ , that is,  $s'_k(t) = s'_k$ , or (2)  $\frac{\partial V_k(\mathbf{s}'(t))}{\partial s_j} \neq \frac{\partial V_j(\mathbf{s}'(t))}{\partial s_j}$  for all  $t \in (0, t^*)$ . In both cases, by selecting  $\tau'$  such that  $s_j(\tau') = s'_j(t)$  for a sufficiently small  $t > 0$ , this special case will be avoided. In the former case this is so because  $\beta_j(s_j(\tau')) \neq \beta_k(s'_k)$ .

Define  $\Theta_k^*(\tau) = \{s'_k : j, k \in \mathcal{I}(s'_j, s'_k)\}$ , which is also finite. Clearly,  $\Theta_k(\tau) \subset \Theta_k^*(\tau)$  (as equal bids imply equal and maximal values but are not necessarily implied by), and the same arguments as above apply to each  $s'_k \in \Theta_k^*(\tau)$ . Note that, since value functions are  $C^1$ , for all  $\hat{\tau}$  sufficiently close to  $\tau$ , and for all  $\hat{s}_k \in \Theta_k(\hat{\tau})$ , the trajectory  $\hat{\mathbf{s}}(t)$ , constructed as above, must coincide with one of the  $\mathbf{s}'(t)$  trajectories (with the appropriate shift in the parameter). So far, we have shown how one can “deal” with each  $(s'_j, s'_k)$  that is not strictly competitive. Since there is at most a finite number of these (with  $s'_k \in \Theta_k^*(\tau)$ ), one can “deal” with all of them (for instance, one-by-one) together and obtain profile  $\tilde{\mathbf{s}}$  close to  $\mathbf{s}$  as desired. (So that, in particular, conditions of Proposition 2 are satisfied, there exists  $U^{\tilde{\mathbf{s}}}$  as in Lemma 5,  $\beta_j$  and  $\beta_k$  are continuous, there are no “special” signals in  $U_j^{\tilde{\mathbf{s}}}$  and  $U_k^{\tilde{\mathbf{s}}}$ , and for each  $s'_k$  such that  $\beta_k(s'_k) = \beta_j(\tilde{s}_j)$  profile  $(s'_k, \tilde{s}_j)$  is strictly competitive for  $\{j, k\}$ .) Reset  $\mathbf{s} = \tilde{\mathbf{s}}$ , and  $U^{\mathbf{s}} = U^{\tilde{\mathbf{s}}}$ .

Now we connect bids to values. For  $s'_j = s_j(0)$  and all  $s'_k \in \Theta_k(0)$  find  $U^{(s_j, s'_k)}$  as in Lemma 5. Then, one can find  $(\tau_-^*, \tau_+^*)$  (for the original  $\mathbf{s}(\tau)$ ) and, for each  $s'_k \in \Theta_k(0)$ , neighborhood  $U^{(s_j, s'_k)} = s_j(\tau_-^*, \tau_+^*) \times U_k^{(s_j, s'_k)}$ , so that  $\beta_k^{-1}(\beta_j(s_j(\tau_-^*, \tau_+^*))) = \cup_{s'_k \in \Theta_k(0)} U_k^{(s_j, s'_k)}$ . By the argument similar to the one in Lemma 6, for each  $\tau \in (\tau_-^*, \tau_+^*)$  we have  $\beta_j(s_j(\tau)) = E_{s'_k \in \Theta_k(\tau)} W_j(s_j(\tau), s'_k)$ . If  $\Theta_k(\tau) = \{s_k(\tau)\}$ , Corollary 2 applies. Otherwise, for each  $s_j(\tau)$  there are at least two different  $s'_k, s''_k \in \Theta_k(\tau)$ .

Consider any such  $s'_j = s_j(\tau)$  and select  $s'_k, s''_k \in \Theta_k(\tau)$ ,  $s'_k < s''_k$ . Then  $W_j(s'_j, s'_k) = W_k(s'_j, s'_k) < W_k(s'_j, s''_k) = W_j(s'_j, s''_k)$ . Let  $\bar{s}_k = \max_{s'_k \in \Theta_k(\tau)} s'_k$ . Since  $f > 0$ ,  $\beta_k(\bar{s}_k) = \beta_j(s'_j) = E_{s'_k \in \Theta_k} W_j(s'_j, s'_k) < W_j(s'_j, \bar{s}_k) = W_k(s'_j, \bar{s}_k)$ . That is, bidder  $k$  with  $\bar{s}_k$  bids less than her value at  $(s'_j, \bar{s}_k)$ . By the same

argument as above, and by the continuity of  $\beta_k$ ,  $\beta_k(\bar{s}_k)$  has to be equal to the expected value of bidder  $k$  with signal  $\bar{s}_k$  over all possible  $s_j''$  such that  $V_k(\bar{s}_k, s_j'') = V_j(\bar{s}_k, s_j'')$ .<sup>26</sup> (Without loss of generality, we can assume that all pairs  $(s_j'', s_k'')$  are strictly competitive, as we can achieve this by a careful selection of  $\tau$  and possibly a slight perturbation of  $s_j(\tau)$  as discussed above.) . Since  $\beta_k(\bar{s}_k) < W_k(s_j', \bar{s}_k)$ , there must exist  $s_j''$ , such that  $W_j(s_j'', \bar{s}_k) = W_k(s_j'', \bar{s}_k) < \beta_k(\bar{s}_k)$ . Let  $\underline{s}_j$  be the lowest of such  $s_j''$ , then  $\beta_j(\underline{s}_j) = \beta_k(\bar{s}_k) > W_j(\underline{s}_j, \bar{s}_k)$ . Repeating the argument for  $\underline{s}_j$ , there must exist  $s_k'' > \bar{s}_k$  such that  $W_j(\underline{s}_j, s_k'') = W_k(\underline{s}_j, s_k'') > \beta_j(\underline{s}_j) = \beta_k(s_k'')$ . Since  $\beta_k(s_k'') = \beta_j(\underline{s}_j) = \beta_k(\bar{s}_k) = \beta_j(s_j')$ , it must be the case (as argued above), that  $V_k(s_j', s_k'') = V_j(s_j', s_k'')$ , which contradicts the fact that  $\bar{s}_k$  is the highest such signal.  $\square$

### A.2.2 Supporting results for value functions

**Corollary 3** Consider interior  $\mathbf{s}$  with  $\#\mathcal{I}(\mathbf{s}) > 2$ , any  $\mathcal{A} \subsetneq \mathcal{I}(\mathbf{s})$  with  $\#\mathcal{A} = n \geq 2$  and bidder  $k \in \mathcal{I}(\mathbf{s}) \setminus \mathcal{A}$ . Suppose GSC is satisfied at  $\mathbf{s}$  for any  $\mathcal{B} \subset \mathcal{I}(\mathbf{s})$  with  $\#\mathcal{B} < n$ . Suppose also that for some  $j \in \mathcal{A}$ ,  $\frac{\partial V_j(\mathbf{s})}{\partial s_k} \neq \frac{\partial V_k(\mathbf{s})}{\partial s_k}$ . Then, GSC is violated at  $\mathbf{s}$  for  $\mathcal{A}$  and  $k$  if and only if (1)  $u_k^{\mathcal{C}} < 0$  and  $\forall j \in \mathcal{A}$ ,  $u_j^{\mathcal{C}} > 0$ , or (2)  $u_k^{\mathcal{C}} > 0$  and  $\forall j \in \mathcal{A}$ ,  $u_j^{\mathcal{C}} < 0$ , where  $\mathcal{C} \equiv \mathcal{A} \cup \{k\}$  and  $\mathbf{u}^{\mathcal{C}}$  is an equal increment vector for the subset  $\mathcal{C}$ .

**Proof** First note that  $\mathbf{u}_{\mathcal{A}}^{\mathcal{A}} \geq \mathbf{0}$  and  $\mathbf{u}_{\mathcal{B}}^{\mathcal{B}} \geq \mathbf{0}$  for all  $\mathcal{B} \subset \mathcal{I}(\mathbf{s})$  with  $\#\mathcal{B} < n$  (this follows from the proof of Lemma 2).

( $\Leftarrow$ ) Suppose that  $u_k^{\mathcal{C}} < 0$ , while  $\mathbf{u}_{\mathcal{A}}^{\mathcal{C}} \gg \mathbf{0}$  (the argument for the other case is similar), and define  $\mathbf{u}' = (-u_k^{\mathcal{C}}, \mathbf{0}_{-k})$ . Since GSC is satisfied for  $\mathcal{B} = \{k\}$ ,  $\mu_k(\mathbf{u}') \geq \max_{i \in \mathcal{A}} \mu_i(\mathbf{u}')$ . If  $\mu_k(\mathbf{u}') > \max_{i \in \mathcal{A}} \mu_i(\mathbf{u}')$ , then GSC is violated for  $k$ ,  $\mathcal{A}$ , and  $\mathbf{u} \equiv \mathbf{u}^{\mathcal{C}} + \mathbf{u}'$  (by construction,  $\mathbf{u}_{\mathcal{A}} \gg \mathbf{0}$ ,  $\mathbf{u}_{-\mathcal{A}} = \mathbf{0}$ ). Otherwise, define  $\mathcal{D} = \arg \max_{i \in \mathcal{A}} \mu_i(\mathbf{u}')$ . We have  $\mathcal{D} \subsetneq \mathcal{A}$ ,  $\mathbf{u}_{\mathcal{D}}^{\mathcal{D}} \geq \mathbf{0}$ , and, by regularity,  $\mu_k(\mathbf{u}^{\mathcal{D}}) < \mu_i(\mathbf{u}^{\mathcal{D}})$  for  $i \in \mathcal{D}$ . Then, for sufficiently small  $\varepsilon > 0$  and vector  $\mathbf{u} \equiv \mathbf{u}^{\mathcal{C}} + \mathbf{u}' - \varepsilon \mathbf{u}^{\mathcal{D}}$  we have: (i)  $\mathbf{u}_{\mathcal{A}} \gg \mathbf{0}$ ,  $\mathbf{u}_{-\mathcal{A}} = \mathbf{0}$ ; (ii) for each  $i \in \mathcal{D}$  and  $j \in \mathcal{A} \setminus \mathcal{D}$ ,  $\mu_i(\mathbf{u}) > \mu_j(\mathbf{u})$ ; (iii)  $\mu_k(\mathbf{u}) > \max_{i \in \mathcal{A}} \mu_i(\mathbf{u})$ , that is GSC is violated for  $k$ ,  $\mathcal{A}$ , and  $\mathbf{u}$ .

( $\Rightarrow$ ) First note that  $u_k^{\mathcal{C}} \neq 0$  (else,  $\mathbf{u}^{\mathcal{C}} = \mathbf{u}^{\mathcal{A}}$  and GSC is satisfied with equality for bidder  $k$  and  $\mathcal{A}$ ) and for all  $i \in \mathcal{A}$ ,  $u_i^{\mathcal{C}} \neq 0$  (else, GSC is violated for a subset with less than  $n$  bidders). We can suppose that all the inequalities in (5) are strict. If not, we can first perturb the Jacobian as we did in Step 2 of the proof of Lemma 2 to eliminate all equalities.

Step 1. We show that  $\mathbf{u}_{\mathcal{C}}^{\mathcal{C}}$  can have either 1 or  $n$  positive components. Partition  $\mathcal{C} = \mathcal{B} \sqcup \mathcal{D}$ , where  $i \in \mathcal{B}$  ( $i \in \mathcal{D}$ ) if  $u_i^{\mathcal{C}} < 0$  ( $u_i^{\mathcal{C}} > 0$ ). Clearly  $\mathcal{D} \neq \emptyset$ . Suppose

<sup>26</sup> Remember, if there exists such  $s_j''$  for at least some  $\mathbf{s}_{-jk} \in \Theta_{-jk}$ ,  $V_k(s_j'', \bar{s}_k, \mathbf{s}'_{-jk}) = V_j(s_j'', \bar{s}_k, \mathbf{s}'_{-jk})$  for all  $\mathbf{s}'_{-jk} \in \Theta_{-jk}$ , and so  $W_k(s_j'', \bar{s}_k) = W_j(s_j'', \bar{s}_k)$ .

that  $\#\mathcal{D} \neq 1, n$  and consider vector  $\mathbf{w}_1(t) = \mathbf{u}^c - t\mathbf{u}^{\mathcal{D}}$ . Since  $\#\mathcal{D} < n$ , we have  $\mathbf{u}_{\mathcal{D}}^{\mathcal{D}} \gg \mathbf{0}$  and  $\forall j \in \mathcal{D}, \forall i \in \mathcal{B}, \mu_j(\mathbf{u}^{\mathcal{D}}) > \mu_i(\mathbf{u}^{\mathcal{D}})$ , thus  $\mu_j(\mathbf{w}_1(t)) < \mu_i(\mathbf{w}_1(t))$  for all  $t > 0$ . Let  $t_1$  be the minimal  $t > 0$  such that  $\mathbf{w}_{1j}(t) = 0$  for some  $j \in \mathcal{D}$ . Let  $\mathcal{E}$  consist of all  $l \in \mathcal{D}$  with  $w_{1l}(t) > 0$  and consider vector  $\mathbf{w}_2(t) = \mathbf{w}_1(t_1) - t\mathbf{u}^{\mathcal{E}}$ . Let  $t_2$  be the minimal  $t > 0$  such that for some bidder  $j \in \mathcal{E}, w_{2j}(t) = 0$ . Again,  $\forall j \in \mathcal{E}, \forall i \in \mathcal{B}, \mu_j(\mathbf{w}_2(t_2)) < \mu_i(\mathbf{w}_2(t_2))$ . By continuing in this manner we eventually obtain vector  $\mathbf{w}_m(t_m)$  such that for all  $j \in \mathcal{D}, w_{mj}(t_m) = 0$ . Fix  $j \in \mathcal{D}$  with  $w_{mj}(0) > 0$ , thus  $\mu_j(\mathbf{w}_m(t_m)) < \mu_i(\mathbf{w}_m(t_m))$  for all  $i \in \mathcal{B}$ . Consider  $\mathbf{w} \equiv -\mathbf{w}_m(t_m)$  and note that  $\mathbf{w}_{\mathcal{B}} = -\mathbf{u}_{\mathcal{B}}^c \gg \mathbf{0}, \mathbf{w}_{-\mathcal{B}} = \mathbf{0}$ . Therefore, GSC is violated for  $\mathcal{B}, j$ , and  $\mathbf{w}$ , which is a contradiction since  $\#\mathcal{B} < n$ .

Step 2. Suppose  $u_k^c > 0$  (the proof for  $u_k^c < 0$  is similar) and  $\mathbf{u}_{\mathcal{A}}^c \not\ll \mathbf{0}$ . Consider  $\mathbf{u}' = (0_k, \mathbf{u}_{-k}^c)$ . Clearly,  $\mathbf{u}' \neq \mathbf{u}^{\mathcal{A}}$ . Since  $\mu_k(\mathbf{u}^c) = \mu_i(\mathbf{u}^c) = 1$  for all  $i \in \mathcal{A}$  and GSC is (strictly) satisfied for  $\mathcal{B} = \{k\}$ , we have  $\mu_k(\mathbf{u}') < \min_{i \in \mathcal{A}} \mu_i(\mathbf{u}')$ . Since GSC is violated for  $\mathcal{A}$  and  $k$ , there exists  $\mathbf{u}$ , with  $\mathbf{u}_{\mathcal{A}} \gg \mathbf{0}$  and  $\mathbf{u}_{-\mathcal{A}} = \mathbf{0}$ , such that  $\mu_k(\mathbf{u}) > \max_{i \in \mathcal{A}} \mu_i(\mathbf{u})$ . Consider  $\mathbf{w}(t) = \mathbf{u} - t\mathbf{u}'$  and note that  $\mu_k(\mathbf{w}(t)) > \max_{i \in \mathcal{A}} \mu_i(\mathbf{w}(t))$  for  $t \geq 0, \mathbf{w}_{-\mathcal{C}}(t) = \mathbf{0}$  for any  $t$ , and  $\mathbf{w}_{\mathcal{A}}(0) \gg \mathbf{0}$ . Since  $u'_i = u_i^c > 0$  for some  $i \in \mathcal{A}$ , there exist the minimal  $t' > 0$ , such that for some  $w_i(t') = 0$  for some  $i \in \mathcal{A}$ . Then, GSC is violated for  $\mathcal{B} = \mathcal{A} \setminus \{i\}, k$ , and  $\mathbf{w}(t')$ , which is a contradiction since  $\#\mathcal{B} < n$ .  $\square$

**Definition 7** For a given  $\mathbf{s}', \mathcal{B} \subset \mathcal{I}(\mathbf{s}')$ , and vector  $\mathbf{x}$  with  $\mathbf{x}_{-\mathcal{B}} = \mathbf{0}$ , define  $\mathbf{y}_{\mathcal{B}} \equiv \mathbf{x} \cdot \nabla \mathbf{V}_{\mathcal{B}}(\mathbf{s}')$ —the derivative of  $\mathbf{V}_{\mathcal{B}}$  along  $\mathbf{x}$ . Define trajectory  $\mathbf{s}^{\mathbf{x}\mathcal{B}}(\tau)$  with  $\mathbf{s}^{\mathbf{x}\mathcal{B}}(0) = \mathbf{s}'$  and  $\mathbf{s}_{-\mathcal{B}}^{\mathbf{x}\mathcal{B}}(\tau) = \mathbf{s}'_{-\mathcal{B}}$  as a solution to the system

$$\mathbf{V}_{\mathcal{B}}(\mathbf{s}^{\mathbf{x}\mathcal{B}}(\tau)) = V\mathbf{1}_{\mathcal{B}} + \tau\mathbf{y}_{\mathcal{B}},$$

where  $V = \max_{j \in \mathcal{N}} V_j(\mathbf{s}')$ . Clearly,  $\left. \frac{d\mathbf{s}_{\mathcal{B}}^{\mathbf{x}\mathcal{B}}(\tau)}{d\tau} \right|_{\tau=0} = \mathbf{x}_{\mathcal{B}}$ .

**Lemma 7** For any proper subset  $\mathcal{B} \subsetneq \mathcal{A}$  there exists  $\mathcal{D} = \mathcal{D}(\mathcal{B}), \mathcal{B} \subseteq \mathcal{D} \subsetneq \mathcal{A}$ , such that for any  $k \in \mathcal{A} \setminus \mathcal{D}, \mathbf{u}^{\mathcal{D}} \cdot \nabla V_k(\mathbf{s}) < 1$ . Also, for any  $\varepsilon > 0$ , there exist vector  $\mathbf{v}^{\mathcal{B}} \geq \mathbf{0}$  such that  $\|\mathbf{v}^{\mathcal{B}} - \mathbf{u}^{\mathcal{B}}\| < \varepsilon, \mathbf{v}_k^{\mathcal{B}} = 0$  for any  $k \in \mathcal{A} \setminus \mathcal{D}, \mathbf{v}^{\mathcal{B}} \cdot \nabla V_i(\mathbf{s}) < 1$  for any  $i \in \mathcal{A} \setminus \mathcal{B}$  and  $\mathbf{v}^{\mathcal{B}} \cdot \nabla V_j(\mathbf{s}) = 1$  for all  $j \in \mathcal{B}$ .

**Proof.** The proof is by induction on the number of bidders in  $\mathcal{B}$ . Define  $\mathcal{C}$  as the set of bidders  $k \in \mathcal{A} \setminus \mathcal{B}$  such that  $\mathbf{u}^{\mathcal{B}} \cdot \nabla V_k(\mathbf{s}) = 1$ . Since GSC is satisfied for  $\mathcal{B}, \mathbf{u}^{\mathcal{B}} \neq \mathbf{u}^{\mathcal{A}}$  and  $\mathcal{C} \neq \mathcal{A} \setminus \mathcal{B}$ . If  $\mathcal{C} = \emptyset$ , then set  $\mathcal{D} \equiv \mathcal{B}$ , and  $\mathbf{v}^{\mathcal{B}} \equiv \mathbf{u}^{\mathcal{B}}$ . Thus, lemma is true for  $\#\mathcal{B} = \#\mathcal{A} - 1 = n - 1$ . Suppose now that lemma is true for all  $\mathcal{B}'$ , with  $\#\mathcal{B}' > \#\mathcal{B}$ . If  $\mathcal{C} \neq \emptyset$  define  $\mathcal{B}' = \mathcal{A} \setminus \mathcal{C}$ , then  $\mathbf{u}^{\mathcal{B}'} \cdot \nabla V_k(\mathbf{s}) < 1$  for any  $k \in \mathcal{B}' \setminus \mathcal{B}$ . Pick  $\mathcal{D} \equiv \mathcal{D}(\mathcal{B}')$ . Consider  $\mathbf{v}^{\mathcal{B}} = \lambda_1 \mathbf{u}^{\mathcal{B}} + (1 - \lambda_1) \mathbf{v}^{\mathcal{B}'}$  with  $\lambda_1 \in (0, 1)$ . When  $\lambda_1 \rightarrow 1, \mathbf{v}^{\mathcal{B}} \rightarrow \mathbf{u}^{\mathcal{B}}$ . By induction, for all  $j \in \mathcal{A} \setminus \mathcal{B}', \mathbf{v}^{\mathcal{B}'} \cdot \nabla V_j(\mathbf{s}) < 1$ . Thus, for all  $j \in \mathcal{A} \setminus \mathcal{B}, \mathbf{v}^{\mathcal{B}} \cdot \nabla V_j(\mathbf{s}) < 1$  as long as  $\lambda_1 \in (0, 1)$ .  $\square$

**Remark 1** As follows from the proof of Lemma 7 we can find a finite sequence  $\mathcal{B} \subsetneq \mathcal{B}' \subsetneq \mathcal{B}'' \subsetneq \dots \subsetneq \mathcal{A}$ , such that  $\mathbf{v}^{\mathcal{B}}(\boldsymbol{\lambda}) = \lambda_1 \mathbf{u}^{\mathcal{B}} + \lambda_2 \mathbf{u}^{\mathcal{B}'} + \lambda_3 \mathbf{u}^{\mathcal{B}''} + \dots$ ,

where  $\sum_i \lambda_i = 1$ ,  $\lambda_i \in (0, 1)$  for all  $i$ , and  $\lambda_1$  is arbitrarily close to 1.

**Lemma 8** For any  $\mathcal{B} \subsetneq \mathcal{A}$  and  $\mathbf{v}^{\mathcal{B}} \equiv \mathbf{v}^{\mathcal{B}}(\boldsymbol{\lambda})$  with  $\lambda_1$  sufficiently close to 1, it is either  $\mathbf{v}^{\mathcal{B}} \cdot \nabla V_1(\mathbf{s}) < \max_{j \in \mathcal{B}} \mathbf{v}^{\mathcal{B}} \cdot \nabla V_j(\mathbf{s})$  or  $\frac{\partial V_1(\mathbf{s}')}{\partial s_1} > \min_{j \in \mathcal{B}} \frac{\partial V_j(\mathbf{s}')}{\partial s_1}$ .

**Proof.** If  $\mathbf{u}^{\mathcal{B}} \cdot \nabla V_1(\mathbf{s}) < 1 = \max_{j \in \mathcal{B}} \mathbf{u}^{\mathcal{B}} \cdot \nabla V_j(\mathbf{s})$ , then also  $\mathbf{v}^{\mathcal{B}} \cdot \nabla V_1(\mathbf{s}) < \max_{j \in \mathcal{B}} \mathbf{v}^{\mathcal{B}} \cdot \nabla V_j(\mathbf{s})$  for  $\lambda_1$  sufficiently close to 1. Otherwise, since  $\#\mathcal{B} < n$ ,  $\mathbf{u}^{\mathcal{B}} \cdot \nabla V_1(\mathbf{s}') = 1$  or  $DV_{\mathcal{C}} \mathbf{u}_{\mathcal{C}}^{\mathcal{B}} = \mathbf{1}_{\mathcal{C}}$ , where  $\mathcal{C} = \mathcal{B} \cup \{1\}$ . SC and regularity imply  $\frac{\partial V_1(\mathbf{s}')}{\partial s_1} > \min_{j \in \mathcal{B}} \frac{\partial V_j(\mathbf{s}')}{\partial s_1}$ .  $\square$

### A.2.3 Proof of Proposition 2 for $\#\mathcal{A} \geq 2$

**Lemma 9** Consider  $\mathbf{s}' = \mathbf{s}(t)$  for an arbitrary  $t \in U_t^0$ ,  $\mathbf{s}(t)$  is the trajectory defined in Step 1 on page 19. For any  $j \in \mathcal{A}$ , there exists  $b_j(s'_j) \equiv \lim_{s_j \downarrow s'_j} \inf \beta_j(s_j)$ , and these limits are equal:  $b_j(s'_j) \equiv b < \infty$ . In addition,  $\beta_1(s'_1) > b$  and for any  $j \in \mathcal{A}$  and  $s_j > s'_j$  sufficiently close to  $s'_j$ ,  $\beta_j(s_j) \geq b$ .

**Proof.** The proof relies on some supporting results about value functions presented in Appendix A.2.2 above. Consider trajectory  $\mathbf{s}^{\mathcal{A}}(\tau) \equiv \mathbf{s}^{\mathbf{u}^{\mathcal{A}}}(\tau)$  with  $\mathbf{s}^{\mathcal{A}}(0) = \mathbf{s}'$ , along which the values of bidders  $\mathcal{A}$  remain equal and  $s'_1$  is fixed (see Definition 7). Because  $\mathcal{A}$  is irreducible,  $\mathbf{u}_{\mathcal{A}}^{\mathcal{A}} \gg \mathbf{0}$  and thus, for any  $j \in \mathcal{A}$ ,  $s_j^{\mathcal{A}}(\tau)$  is strictly increasing (for  $\tau$  close to 0). Since GSC is violated for  $\mathcal{A}$  and 1,  $V_1(\mathbf{s}^{\mathcal{A}}(\tau)) > \max_{j \in \mathcal{A}} V_j(\mathbf{s}^{\mathcal{A}}(\tau))$  for any sufficiently small  $\tau > 0$ . Therefore,

$$\beta_1(s'_1) > \min_{j \in \mathcal{A}} \beta_j(s_j^{\mathcal{A}}(\tau)). \quad (\text{A.5})$$

In turn, by continuity of  $\mathbf{V}$ , for any  $s_1$  sufficiently close to  $s'_1$  and  $s_1 < s'_1$ ,

$$\beta_1(s_1) > \min_{j \in \mathcal{A}} \beta_j(s_j^{\mathcal{A}}(\tau)). \quad (\text{A.6})$$

Define  $b_{\mathcal{B}} \equiv \min_{i \in \mathcal{A}} b_i(s'_i)$ ,  $\mathcal{B} = \arg \min_{i \in \mathcal{A}} b_i(s'_i)$ , and  $b_{-\mathcal{B}} \equiv \min_{i \in \mathcal{A} \setminus \mathcal{B}} b_i(s'_i)$ . (It is possible that  $b_{\mathcal{B}} = \infty$  and/or  $b_{-\mathcal{B}} = \infty$ . In what follows,  $a > b$  together with  $b = \infty$  implies  $a = \infty$ .) By definition,  $b_{\mathcal{B}} < \infty$  when  $\#\mathcal{B} < n = \#\mathcal{A}$ . We show by induction on  $\#\mathcal{B}$  that unless  $\#\mathcal{B} = n$  efficiency is violated.

Step 1. Suppose  $\#\mathcal{B} = 1$ , let  $\mathcal{B} = \{j\}$  and fix  $\varepsilon = (b_{-j} - b_j)/2$ . There exists  $\delta_\tau > 0$ , such that for all  $\tau \in (0, \delta_\tau)$  and  $i \in \mathcal{A} \setminus \{j\}$ ,  $\beta_i(s_i^{\mathcal{A}}(\tau)) > b_{-j} - \varepsilon$  and (A.5) holds. Consider trajectory  $\mathbf{s}^*(r) = \mathbf{s}' + r\mathbf{v}^j$ , where  $\mathbf{v}^j \equiv \mathbf{v}^j(\boldsymbol{\lambda})$  is defined in Lemma 7 (see also Remark 1), and  $\lambda_1$  is sufficiently close to 1 so that Lemma 8 holds. Then, there exists  $\delta_r > 0$ , so that for all  $r \in (0, \delta_r)$  and for all  $i \in \mathcal{A}$ ,  $s_i^*(r) \in s_j^{\mathcal{A}}[0, \delta_\tau)$  (that is,  $s_i^*(r) = s_i^{\mathcal{A}}(\tau)$  for some  $\tau \in [0, \delta_\tau)$ ).

Consider a sequence  $s_{j_m} \downarrow s'_j$ , for which  $\beta_j(s_{j_m}) \rightarrow b_j(s'_j)$  and, for all  $m$ ,

$\beta_j(s_{jm}) < b_j + \varepsilon$  and  $r_m \in (0, \delta_r)$ , where  $s_j^*(r_m) = s_{jm}$ . Then, by (A.5),

$$\beta_1(s'_1) > \beta_j(s_j^*(r_m)) = \min_{j \in \mathcal{A}} \beta_j(s_j^A(\tau_m)), \quad (\text{A.7})$$

where  $\tau_m$  is defined by  $s_j^A(\tau_m) = s_j^*(r_m)$ . Also, similarly to (A.6), there exists  $\delta_1 > 0$ , such that for any  $s_1 \in (s'_1 - \delta_1, s'_1)$ ,

$$\beta_1(s_1) > \beta_j(s_j(r_m)). \quad (\text{A.8})$$

Since  $\#\mathcal{B} < n$ , by continuity and Lemma 7, for sufficiently large  $m$  (and so for sufficiently small  $r_m > 0$ ),  $\{j\} \subset \mathcal{I}(\mathbf{s}^*(r_m)) \subset \{j, 1\}$ . By continuity and Lemma 8, for sufficiently small  $r_m > 0$ , there exists  $\delta'_1(r_m) > 0$ , such that for any  $s_1 \in (s'_1 - \delta'_1(r_m), s'_1)$ ,

$$\{j\} = \mathcal{I}(s_1, \mathbf{s}_A^*(r_m)). \quad (\text{A.9})$$

Pick  $s_1$  such that both (A.8) and (A.9) hold. Then, even if we slightly increase the signals of those  $i \in \mathcal{A}$  for whom  $s_i^*(r_m) = s'_i$ , by continuity,  $\{j\} = \mathcal{I}(\mathbf{s})$  at the so obtained profile  $\mathbf{s}$ . Since  $\beta_i(s_i) > b_{-j} - \varepsilon$  for any  $s_i \in (s'_i, s_i^A(\delta_\tau))$ ,  $\beta_j(s_{jm}) = \beta_j(s_j) = \min_{i \in \mathcal{A}+1} \beta_i(s_i)$ . Thus, we have reached a contradiction as  $j$  has the highest value but drops out the first.

Step 2. Here we show that for any  $j \in \mathcal{A}$  with  $s_j > s'_j$  and sufficiently close to  $s'_j$ ,  $\beta_j(s_j) \geq b_{\mathcal{B}}$ . For each  $j \in \mathcal{A}$  pick a trajectory  $\mathbf{s}^j(r) = \mathbf{s}' + r\mathbf{v}^j(\boldsymbol{\lambda}^j)$ , where  $\mathbf{v}^j(\boldsymbol{\lambda}^j)$  satisfies Lemmata 7 and 8 (see also Remark 1), and so that  $\forall j \in \mathcal{A}$ ,  $\lambda_1^j \in (1 - \delta_\lambda, 1)$ , where  $\delta_\lambda > 0$  and is arbitrarily close to 0. Then, there exists  $\delta_r > 0$ , so that for all  $r \in (0, \delta_r)$  and for all  $i, j \in \mathcal{A}$  we have: (1)  $s_i^j(r) = s_i^A(\tau)$  for some  $\tau \in [0, \delta_\tau]$ ; and (2)  $\{j, 1\} \supset \mathcal{I}(\mathbf{s}^j(r))$  and  $\{j\} = \mathcal{I}(s_1, \mathbf{s}_A^j(r))$  for any  $s_1 \leq s'_1$  sufficiently close to  $s'_1$  (the latter may depend on particular  $r$  and  $j$ ).

Suppose that there exists  $j \in \mathcal{A}$  with  $\beta_j(s_j) < b_{\mathcal{B}}$  for  $s_j > s'_j$  arbitrarily close to  $s'_j$ . Pick such  $s_j$  so that  $s_j = s_j^j(r_j)$  and  $r_j \in (0, \delta_r)$ , let  $\varepsilon = b_{\mathcal{B}} - \beta_j(s_j) > 0$ . Then,  $\{j, 1\} \supset \mathcal{I}(\mathbf{s}^j(r))$  and, as in Step 1, by slightly reducing  $s'_1$  and slightly increasing signals of all  $i \in \mathcal{A}$  with  $s_i^j(r_j) = s'_i$ , we obtain profile  $\mathbf{s}$  with  $\{j\} = \mathcal{I}(\mathbf{s})$ . Thus, there exists  $i \in \mathcal{A}+1$  with  $\beta_i(s_i) < \beta_j(s_j)$ . If  $\beta_1(s_1)$  is always the smallest no matter how slight the decrease in  $s_1$  is, then  $\lim_{s_1 \uparrow s'_1} \sup \beta_1(s_1) \leq b_{\mathcal{B}} - \varepsilon$ , contradicting (A.6). Thus, there exist  $i \in \mathcal{A}$  with  $s_i > s'_i$ , such that  $\beta_i(s_i) < \beta_j(s_j) = b_{\mathcal{B}} - \varepsilon$  and  $r_i$ , defined as  $s_i^i(r) = s_i$ , satisfies  $r_i < Cr_j$ . By choosing  $\delta_\lambda$  as close to 1 as necessary, we can make  $C$  as close to zero as necessary. It suffices to have  $C < 1$ . By repeating this procedure we either find a contradiction that involves bidder 1 or find  $i \in \mathcal{A}$  and a converging sequence of  $r_{im} \downarrow 0$ , such that  $s_{im} = s_i^i(r_{im}) \downarrow s'_i$  and  $\beta_i(s_{im}) < b_{\mathcal{B}} - \varepsilon$  for any  $m$ . But then,  $b_i(s'_i) \leq b_{\mathcal{B}} - \varepsilon$ .

Step 3. Suppose now that  $\#\mathcal{B} = k \geq 2$ . We show that we can find a trajectory  $\mathbf{s}(\rho)$  along which bidders from  $\mathcal{B}$  have the highest value and are dropping si-

multaneously for almost all  $\rho$ . Then, as in Step 1, even after slightly increasing signals of bidders from  $\mathcal{A} \setminus \mathcal{B}$ , there exists  $i \in \mathcal{A} \setminus \mathcal{B}$  who drops earlier. Thus, for some  $i \in \mathcal{A} \setminus \mathcal{B}$ ,  $b_i(s'_i) \leq b_{\mathcal{B}}$ .

Formally, consider trajectory  $\mathbf{s}^{\mathcal{B}}(\rho) = \mathbf{s}^{\mathcal{V}^{\mathcal{B}\mathcal{D}}}(\rho)$ , for  $\rho \geq 0$  and  $\mathbf{s}^{\mathcal{B}}(0) = \mathbf{s}'$ , defined for subset  $\mathcal{D}$  from Lemma 7. (It is possible that  $v_j^{\mathcal{B}} = 0$  for some  $j \in \mathcal{B}$ , thus  $s_j^{\mathcal{B}}(\rho)$  is not necessarily increasing.) Define  $b_j(\rho) \equiv b_j(s_j^{\mathcal{B}}(\rho))$ , clearly,  $\lim_{\rho \rightarrow 0} b_j(\rho) = b_j(s'_j)$  for all  $j \in \mathcal{B}$ . By construction, since GSC is satisfied for  $\mathcal{D}$  and 1, and by Lemmata 7 and 8, for sufficiently small  $\rho > 0$ ,  $\mathcal{I}(\mathbf{s}^{\mathcal{B}}(\rho)) \subset \mathcal{B} \cup \{1\}$  and, once we slightly decrease the signal of bidder 1, all the bidders with the highest value belong to  $\mathcal{B}$ . Also, for sufficiently small  $\rho$ ,  $\max_{j \in \mathcal{B}} b_j(\rho) < b_{-\mathcal{B}}$ .

Proceeding from the contrary, by the arguments similar to the one in Step 1 applied to  $\mathbf{s} = \mathbf{s}^{\mathcal{B}}(\rho)$  for a sufficiently small  $\rho > 0$ , and by induction (on the size of a subset of  $\mathcal{B}$ ), we have that  $b_j(\rho) = b_{\mathcal{B}}(\rho)$  for all  $j \in \mathcal{B}$ . In general, as long as we stay sufficiently close to  $\mathbf{s}'$ , we can, by slightly moving in appropriate directions away from  $\mathbf{s}'$  and possibly in several steps, separate bidders  $\mathcal{A}$  in any given order. Thus, whenever two or more bidders from  $\mathcal{A}$  have equal and maximal values, the limits of their bids from the right have to be equal.

Similarly, by the argument as in Step 2,  $\beta_j(s_j) \geq b_{\mathcal{B}}(\rho)$  for any  $j \in \mathcal{A}$  with  $s_j > s_j^{\mathcal{B}}(\rho)$  and sufficiently close to  $s_j^{\mathcal{B}}(\rho)$ . Therefore, since  $s_j^{\mathcal{B}}(\rho)$  is strictly increasing for some  $j \in \mathcal{B}$ ,  $b_{\mathcal{B}}(\rho)$  is weakly increasing.

There exists  $\delta_\rho > 0$ , so that for all  $\rho \in (0, \delta_\rho)$  and any  $j \in \mathcal{B}$ ,  $\beta_j(s_j^{\mathcal{B}}(\rho)) \leq b_{\mathcal{B}}(\rho)$ . (Otherwise, for any  $\delta_\rho > 0$  we can find  $\rho \in (0, \delta_\rho)$  and  $j \in \mathcal{B}$ , such that  $\beta_j(s_j^{\mathcal{B}}(\rho)) > b_{\mathcal{B}}(\rho)$ . Then, after fixing  $s_j^{\mathcal{B}}(\rho)$  and  $\beta_j(s_j^{\mathcal{B}}(\rho))$ , by induction and repeating the above arguments for  $\mathcal{B}' = \mathcal{B} \setminus \{j\}$  and  $\mathcal{A}' = \mathcal{A} \setminus \{j\}$ , we will reach a contradiction.) In turn, since  $b_{\mathcal{B}}(\rho)$  is monotone and for all  $j$ ,  $\beta_j(s_j) \geq b_{\mathcal{B}}(\rho)$  for  $s_j > s_j^{\mathcal{B}}(\rho)$  (locally), we have that  $\rho$  is a discontinuity point for  $b_{\mathcal{B}}(\rho)$  if  $\beta_j(s_j^{\mathcal{B}}(\rho)) < b_{\mathcal{B}}(\rho)$  for some  $j \in \mathcal{B}$ . Since a monotone function can have only a countable number of discontinuity points, we have that for almost all  $\rho$ , for all  $j$ ,  $\beta_j(s_j^{\mathcal{B}}(\rho)) = b_{\mathcal{B}}(\rho)$ .

Now, we add bidder 1 into the picture. First, suppose that for some  $j \in \mathcal{B}$  there exists  $\delta_\rho > 0$ , such that  $s_j^{\mathcal{B}}(\rho) = s'_j$  for  $\rho \in (0, \delta_\rho)$ . Then,  $b_{\mathcal{B}}(\rho) = b_{\mathcal{B}}$ , and so for any  $i \in \mathcal{B}$  with  $s_i^{\mathcal{B}}(\rho)$  strictly increasing,  $\beta_i(s_i^{\mathcal{B}}(\rho)) = b_{\mathcal{B}}$  for all  $\rho \in (0, \delta_\rho)$ . Consider  $\mathbf{s}^{\mathcal{A}}(\tau)$ . If  $\tau > 0$  is sufficiently small, the bids of bidders  $\mathcal{A} \setminus \mathcal{B}$  are separated away from  $b_{\mathcal{B}}$ , for each  $j \in \mathcal{B}$ ,  $\beta_j(s_j^{\mathcal{A}}(\tau)) \geq b_{\mathcal{B}}$ , and for some  $i \in \mathcal{B}$ ,  $\beta_i(s_i^{\mathcal{A}}(\tau)) = b_{\mathcal{B}}$ . Therefore, from (A.5),  $\beta_1(s'_1) > b_{\mathcal{B}} = \min_{j \in \mathcal{A}} \beta_j(s_j^{\mathcal{A}}(\tau))$ . Let  $\mathcal{B}'$  consist of  $i \in \mathcal{B}$ , for whom  $\beta_i(s_i) = b_{\mathcal{B}}$  in the right neighborhood of  $s'_i$ . Consider a trajectory  $\mathbf{s}^*(r) = \mathbf{s}' + r\mathbf{v}^{\mathcal{B}'}$ . Along this trajectory, the set of bidders with the highest value is a subset of  $\mathcal{B}' \cup \{1\}$ . By continuity and Lemma 8, for a sufficiently small  $r > 0$ , (A.6) holds as well, and, once  $s'_1$  is slightly reduced,

all the bidders with the highest value belong to  $\mathcal{B}'$ . After slightly increasing the signal of each  $j \in \mathcal{A}$  with  $s_j^*(r) = s'_j$ , we obtain profile  $\mathbf{s}$ , at which bidders  $\mathcal{B}'$  drop out simultaneously at  $b_{\mathcal{B}} = \min_{j \in \mathcal{A}+1} \beta_j(s_j)$ —a contradiction.

In the remaining case, for all  $j \in \mathcal{B}$ ,  $s_j^{\mathcal{B}}(\rho)$  is strictly increasing and so  $\beta_j$  is monotone in the right neighborhood of  $s'_j$ . Since  $\mathbf{u}_{\mathcal{A}}^{\mathcal{A}} \gg \mathbf{0}$ , for any small  $\tau > 0$  and for each  $j \in \mathcal{B}$ , let  $\rho_j$  be the solution to  $s_j^{\mathcal{B}}(\rho_j) = s_j^{\mathcal{A}}(\tau)$  and  $\rho' \equiv \min_{j \in \mathcal{B}} \rho_j$ . For any  $\varepsilon > 0$  there exist  $\delta_\tau > 0$ , such that for any  $\tau \in (0, \delta_\tau)$ , we have: (i) for any  $i \in \mathcal{A} \setminus \mathcal{B}$ ,  $\beta_i(s_i^{\mathcal{A}}(\tau)) > b_{-\mathcal{B}}(\mathbf{s}') - \varepsilon/2$ ; (ii) for all  $j \in \mathcal{B}$ ,  $\rho_j$  is sufficiently small so that  $|b_{\mathcal{B}}(\rho) - b_{\mathcal{B}}| < \varepsilon/2$  and the above results hold. (That is, in particular: (1) the bidders from  $\mathcal{B}$  have the highest value at  $\mathbf{s}^{\mathcal{B}}(\rho)$ , (2)  $b_{\mathcal{B}}(\rho)$  is weakly increasing and, for all  $j$ ,  $\beta_j(s_j^{\mathcal{B}}(\rho)) \leq b_{\mathcal{B}}(\rho)$ ; and (3) starting from  $\mathbf{s}^{\mathcal{B}}(\rho')$ , once  $s'_1$  is slightly reduced, all the bidders with the highest value belong to  $\mathcal{B}$ .)

Pick any  $\tau \in (0, \delta_\tau)$  such that  $b_{\mathcal{B}}(\rho)$  is continuous at  $\rho'$ . Then, consider  $i \in \mathcal{B}$  with  $\rho_i = \rho'$ . From (A.6), for any  $s_1 < s'_1$  and sufficiently close to  $s'_1$ , we have

$$\beta_1(s_1) > \min_{j \in \mathcal{B}} \beta_j(s_j^{\mathcal{A}}(\tau)) = \min_{j \in \mathcal{B}} \beta_j(s_j^{\mathcal{B}}(\rho_j)) = \beta_i(s_i^{\mathcal{B}}(\rho')) = b_{\mathcal{B}}(\rho').$$

Then, starting from  $\mathbf{s}^{\mathcal{B}}(\rho')$ , by reducing slightly  $s'_1$  and increasing slightly  $s'_j$  for each  $j \in \mathcal{A}$  with  $s_j^{\mathcal{B}}(\rho') = s'_j$ , we obtain profile  $\mathbf{s}$ , at which  $\mathcal{I}(\mathbf{s}) \subset \mathcal{B}$ , but bidders  $\mathcal{B}$  exit first simultaneously at  $b_{\mathcal{B}}(\rho')$ —a contradiction.

Step 4. We have shown that  $\#\mathcal{B} = n$ , and so  $\mathcal{B} = \mathcal{A}$ . Let  $b \equiv b_{\mathcal{A}}$ . Since for all  $j \in \mathcal{A}$ ,  $\beta_j(s_j) \geq b(\mathbf{s}')$  for all  $s_j > s'_j$  close to  $s'_j$ , from (A.5) we have  $\beta_1(s'_1) > b$ .

It remains to be shown that  $b < \infty$ . If  $b = \infty$ , then for each  $j \in \mathcal{A}+1$  there exists an interval of signals with  $\beta_j(s_j) = \infty$ . Then, at a profile with such signals, each bidder's payoff is  $-\infty$ , which cannot happen in equilibrium since instead each bidder can exit at  $p = 0$  and assure herself the payoff of 0.  $\square$

**Corollary 4** *If for some  $j \in \mathcal{A}$ ,  $\beta_j(s_j(t)) \neq b(t)$ , then  $t$  is a discontinuity point for  $b(t)$ .*

**Proof.** If for some  $j \in \mathcal{A}$ ,  $\beta_j(s_j(t)) > b(t)$ , then by the argument similar to the one in Step 3 of the proof of Lemma 9, considering  $\mathcal{A}' = \mathcal{A} \setminus \{j\}$ , we can find a profile  $\mathbf{s}$ , at which all the bidders from  $\mathcal{A}'$  exit simultaneously prior to bidder 1 and  $j$ , while all the bidders with the highest value belong to  $\mathcal{A}'$ .

Monotonicity of  $b(t)$  is established in Step 3 of the proof of Proposition 2. From Lemma 9 it follows that for all  $j \in \mathcal{A}$ , whenever  $s_j(t') > s_j(t)$ ,  $\beta_j(s_j(t')) \geq b(t)$ , for  $t$  and  $t'$  from the considered neighborhood  $U_t^0$ . Therefore, if for some  $j \in \mathcal{A}$ ,  $\beta_j(s_j(t)) < b(t)$ , then  $b(t'') \leq \beta_j(s_j(t))$  whenever  $s_j(t'') < s_j(t)$ , so  $t$  is a discontinuity point for  $b(t)$ .  $\square$

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