# 14.773 Political Economy of Institutions and Development.
# Lecture 5. Dynamic Voting with Changing Constituencies

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# Introduction

- Markov Perfect Equilibria different from myopic rules because they take into account the effect of current votes on future political decisions.
- These issues are more salient and important when current political decisions affect the d*istribution of political power* in the future.
- The set of issues that arise here are very similar to those that will be central when we think about endogenous institutions.
- Thus useful to start considering more general dynamic voting models.

# Dynamic Voting in Clubs

- Let us start with a model due to Robert's (1999).
- Voting directly over club size (utilities directly from club size).
- Relatively parsimonious model, but it gets quickly complicated.
- Nevertheless, some important insights can be obtained.
- We will see both later in the lecture and when we study endogenous institutions later in the class how similar insights arise in different settings.
- Key issue: what type of structure we should impose on dynamic models so that they are tractable, while capturing real-world relevant phenomena?

# Environment I

- Consider an economy consisting of a finite group $X = \{1, 2...\bar{x}\}$.
- To make the model tractable, it is assumed that there is an actual seniority system whereby if the voting population is of size $x$, it includes individuals $\{1, 2, ..., x\}$, i.e., lower index individuals are always included before higher index individuals.
- Let the set of potential clubs be denoted by $\mathcal{X}$ (these are sets of the form $\{1\}$, $\{1, 2\}$, etc.).
- Let us denote the size of the voting population at time $t$ by $x_t$ and assume that the instantaneous utility of individual $\xi$ when the size of the (voting) club is $x$ is given by

$$u(x, \xi).$$

- In terms of more micro models, this instantaneous utility function incorporates what the utility of individual $\xi$ will be when tax policies are determined by a club of $x$ individuals.

# Environment II

- Given this instantaneous utility function, the expected utility of individual $\xi$ at time $t = 0$ is given by:

$$U_0 = \mathbb{E}_0 \sum_{t=0}^{\infty} \delta^t u(x_t, \xi), \tag{1}$$

  where $0 < \delta < 1$ is the discount factor.
- The key assumption that will simplify the analysis is the following.

**Assumption (Strict Increasing Differences)** For all $x > x'$, $\xi > \xi'$, we have

$$u(x, \xi) - u(x', \xi) > u(x, \xi') - u(x', \xi').$$

- Higher ranked individuals included later in the club than lower-ranked individuals, but also have preference towards larger groups.
  - This would make sense, for example, when we think of larger "franchises" as leading to higher taxes, and higher taxes being more damaging to richer individuals.

# Environment III

- Two points are noteworthy:

1. Increasing Differences has an obvious parallel to the single-crossing property we introduced earler. You should verify that Increasing Differences is a form of a single-crossing assumption. We will see that this assumption will play a similar role here.
2. Note that this assumption does not mean that individuals like or dislike larger clubs. It is possible, for example, for all individuals to dislike larger clubs (franchises). The assumption simply compares the relative preferences of individuals of different ranks.

# Environment IV

- Let us consider Markov transition rules for analyzing how the size of the club changes over time.
- A Markov transition rule is denoted by $y$ such that

$$y : X \to X.$$

- A transition rule is useful because it defines the path of $x$ recursively such that for all $t$, i.e.,

$$x_{t+1} = y(x_t).$$

- Why Markov?
- If there is an $x$ such that $x = y(x)$, then $x$ is a *steady state* of the system.
- In general, we can consider both deterministic and stochastic transition rules $y(\cdot)$. Later to establish existence of equilibrium, it will be important to allow for mixed strategies and thus stochastic rules.

# Recursive Representation

- The utility of individual $\xi$ under Markov transition rule $y(\cdot)$, starting at $x$, is given by $V(x, \xi, y(\cdot))$ such that

$$V(x, \xi, y(\cdot)) = u(x, \xi) + \delta V(y(x), \xi, y(\cdot)), \tag{2}$$

  where it is important to note that $y(\cdot)$ denotes the entire function.
- Denoting the set of potential transition functions by $\mathcal{Y}$, the value function is:

$$V : X \times X \times \mathcal{Y} \to \mathbb{R}$$

- In the case where $y(\cdot)$ is stochastic, then (1) is assumed to be the von Neumann-Morgenstern utility function, and (2) should be written as:

$$V(x, \xi, y(\cdot)) = u(x, \xi) + \delta \mathbb{E} V(y(x), \xi, y(\cdot)),$$

- **Difficulty (fairly common):** preferences over club size, given by $V$, are conditional on the transition rule $y(\cdot)$.

# Markov Voting Equilibrium

## Definition

Given any transition rule $y^*(\cdot)$, for each $x \in X$, let $Y^*(x)$ be the set of $y$ such that for all $z \in X$:

$$\begin{aligned} & \#\{\xi : \xi \le x \ \& \ V(y,\xi,y^*) > V(z,\xi,y^*)\} \\ \ge \ & \#\{\xi, \xi \le x \ \& \ V(y,\xi,y^*) < V(z,\xi,y^*)\} \end{aligned} \tag{3}$$

If $y^*(x) \in Y^*(x)$ for all $x$ then $y^*(\cdot)$ is a **Markov Voting Equilibrium (MVE)**.

- This condition says that, given that $y^*(\cdot)$ is followed in the future, no club size defeats $y^*(x)$ in pairwise majority voting as the choice for the next period—$y^*(x)$ is the Condorcet winner for the club of size $x$.
    - If $y^*(\cdot)$ is stochastic then (3) must hold for all realizations of $y^*(\cdot)$ that occur with non-zero probability.

# Markov Voting Equilibrium: Interpretation

- Voting for the state that maximizes $V$ is a weakly dominant strategy for each member [as usual, there are other, degenerate Nash equilibria where, for instance, everybody votes for the *status quo* because no single member can disrupt such an outcome, but such possibilities are ruled out by assumption].
- Note that as the electorate at $t+1$ will be $\{1, ...y^*(x_t)\}$, the majority winner chosen by this group will in general be different to that chosen by $\{1, ...x_t\}$ and so $y^*(y^*(x_t))$ may differ from $y^*(x_t)$.

# Does the Median Voter Make Sense Here?

- Median-voter rule: for every club size, an individual with median seniority chooses the club size for the next period.
- **Key result:** single-crossing type assumption sufficient to lead to median voter rules.
- More explicitly:
  - A *median voter* is a function $m(\cdot)$ such that, for club $x$, $m(x)$ is $(x+1)/2$ when $x$ is odd and either $x/2$ or $(x/2)+1$ when $x$ is even.
  - The group of individuals both below and above the median voter, including the median voter, constitutes a weak majority.

# Median Voter Rule

**Definition**

Given any transition rule $\tilde{y}(\cdot)$, for each $x \in X$, let $\tilde{Y}(x)$ be the set of $y$ such that for all $z \in X$:

$$\begin{array}{lll} x \text{ odd:} & & V(y, m(x), \tilde{y}(\cdot)) \geq V(z, m(x), \tilde{y}(\cdot)) \\ x \text{ even:} & \textit{either} & V(x, m(x), \tilde{y}(\cdot)) \geq V(z, m(x), \tilde{y}(\cdot)) \text{ for each } m(x) \\ & \textit{or} & V(x, m(x), \tilde{y}(\cdot)) > V(z, m(x), \tilde{y}(\cdot)) \text{ for some } m(x) \end{array} \tag{4}$$

If $\tilde{y}(x) \in \tilde{Y}(x)$ for all $x$ then $\tilde{y}(\cdot)$ is a **Median Voter Rule**.

# Characterization of Equilibrium I

- Equilibrium analysis will characterize both Markov voting equilibria and median voter equilibria.
- Both of these have the difficulty that they involve the value functions of voters, and the value functions are endogenously determined as a function of future votes (equilibria).
- In the absence of this dynamic linkage, characterization of the equilibrium would be straightforward.
- For example, in the hypothetical case where a club of size $x$ would vote once for a change in the size of the club and there will be no future vote, the median voter choice $\tilde{y}$ could be easily determined as:

$$\begin{array}{lll} x \text{ odd:} & & u(\tilde{y}, m(x)) \geq u(z, m(x)) \\ x \text{ even:} & \textit{either} & u(\tilde{y}, m(x)) \geq u(z, m(x)) \text{ for each } m \\ & \textit{or} & u(\tilde{y}, m(x)) > u(z, m(x)) \text{ for some } m. \end{array} \tag{5}$$

- Such a $\tilde{y}$ exists since $X$ is finite: trivially when $x$ is odd, and when $x$ is even, let $\tilde{y}$ be a best choice for $m(x) = (x/2) + 1$ from among the best outcomes for $m(x) = x/2$.

# Characterization of Equilibrium II

- Assume now that $\tilde{y} > z$ for some $z$ (the alternative case $z > \tilde{y}$ is treated similarly).
- Applying the strict increasing differences condition, we have

$$u(\tilde{y}, \xi) > u(z, \xi) \tag{6}$$

  for all $\xi > (x+1)/2$, $x$ odd, and $\xi > x/2$, $x$ even—both median voters cannot be indifferent when $x$ is even.

- Thus

$$\#\left\{\xi : \xi \leq x \;\&\; u(\tilde{y}, \xi) > u(z, \xi)\right\} \geq \#\xi, \xi \leq x \;\&\; u(\tilde{y}, \xi) < u(z, \xi)\right\}, \tag{7}$$

  and $\tilde{y}$ would indeed be a majority (median voter) solution.

- The idea of the approach pursued here is to try to extend these notions to the dynamic case. In particular, we would like to extend the single crossing features to the dynamic voting situations.

# Characterization of Equilibrium III

**Lemma**

*Given any $x \in X$, let $(y_0, ..., y_t, ...)$ be a sequence such that $y_t \in X$ and $y_t \geq x$ for all $t$, $y_t > x$ for some $t$. If an individual $\xi$ weakly prefers a constant $x$ to the stream $\{y_t\}$, then there is strict preference for all $\xi'$, such that $\xi' < \xi$*

$$\begin{aligned} \sum_{t=0}^{\infty} \delta^t u(x,\xi) &\geq \sum_{t=0}^{\infty} \delta^t u(y_t,\xi) \\ &\Longrightarrow \sum_{t=0}^{\infty} \delta^t u(x,\xi') > \sum_{t=0}^{\infty} \delta^t u(y_t,\xi') \end{aligned} \tag{8}$$

*The same conclusion also follows for any $\xi' > \xi$ if $y_t \in X$ and $y_t < x$ for all $t$, $y_t < x$ for some $t$.*

# Proof of Lemma

- Consider the case where $y_t \geq x$ and $\xi' < \xi$.
- Using the assumption on the strict increasing differences, this implies

  $$u(y_t, \xi) - u(x, \xi) \geq u(y_t, \xi') - u(x, \xi')$$

  with strict inequality for some $t$.
- Weighting by $\delta^t$ and summing over all $t$ gives the strict inequality in the second line of (8).
- The proof of the converse case is analogous.

# Key Theorem

## Theorem

*Consider a dynamic path* $\{y_t\}$ *generated by a non-stochastic MVE, i.e.,* $y^* : y_{t+1} = y^*(y_t)$ *for all* $t \geq 0$ *or by a median voter rule* $\tilde{y} : y_{t+1} = \tilde{y}(y_t)$ *for all* $t \geq 0$*. Then the following types of cycles are not possible: (i)* $y_0 > y_1 \leq y_t$*, for all* $t \geq 2$ *and with* $y_1 < y_\tau$ *for some* $\tau \geq 2$*, or (ii)* $y_0 < y_1 \geq y_t$*, for all* $t \geq 2$ *with* $y_1 > y_t$ *and for some* $\tau \geq 2$*.*
*The same conclusions hold for stochastic MVE and median voter rules, i.e., if* $y^*$ *is stochastic, then a sample path with* $y_0 > y_1 \leq y_t$*, for all* $t \geq 2$ *and with* $y_1 < y_\tau$ *for some* $\tau \geq 2$ *has zero probability.*

# Proof of Theorem I

- Suppose, to obtain a contradiction, that under some MVE, $y^*$, there is a dynamic path with $y_0 > y_1 \leq y_t, t \geq 2$ and $y_1 < y_\tau$ for some $\tau \geq 2$ (this covers the "*and*" case; the "*or*" case is treated similarly).
- **Key observation:** since

$$V(y_1, \xi, y^*) = u(y_1, \xi) + \delta V(y^*(y_1), \xi, y^*) \tag{9}$$

We have

$$V(y_1, \xi, y^*) \overset{\leq}{\geq} V(y_2, \xi, y^*) \tag{10}$$

*if and only if*

$$\sum_{t=2}^{\infty} \delta^{t-1} u(y_1, \xi) \overset{\leq}{\geq} \sum_{t=2}^{\infty} \delta^{t-1} u(y_t, \xi)$$

# Proof of Theorem II

- Why? Think *weighted averages*. For example,

$$\begin{aligned} V(y_1, \xi, y^*) &= u(y_1, \xi) + \delta V(y^*(y_1), \xi, y^*) \\ &> V(y^*(y_1), \xi, y^*) \\ &= \frac{1}{\delta} \sum_{t=1}^{\infty} \delta^{t-1} u(y_t, \xi) \end{aligned}$$

$\Rightarrow$

$$u(y_1, \xi) + \sum_{t=1}^{\infty} \delta^{t-1} u(y_t, \xi) > \frac{1}{\delta} \sum_{t=1}^{\infty} \delta^{t-1} u(y_t, \xi)$$

$\Rightarrow$

$$\frac{\delta}{1-\delta} u(y_1, \xi) \equiv \sum_{t=1}^{\infty} \delta^{t-1} u(y_1, \xi) > \sum_{t=1}^{\infty} \delta^{t-1} u(y_t, \xi).$$

# Proof of Theorem III

- If $y_t \geq y_1$ for all $t \geq 2$, with the inequality being strict for some $t$, Lemma above implies that there must exist $\xi^*, \xi^{**}, \xi^* \leq \xi^{**}$ such that

$$\begin{array}{lcl} \xi \leq \xi^* & \Longleftrightarrow & V(y_1, \xi, y^*) > V(y_2, \xi, y^*) \\ \xi^* < \xi \leq \xi^{**} & \Longleftrightarrow & V(y_1, \xi, y^*) = V(y_2, \xi, y^*) \\ \xi^{**} < \xi & \Longleftrightarrow & V(y_1, \xi, y^*) < (y_2, \xi, y^*) \end{array} \tag{11}$$

- Moreover, since the increasing difference condition is strict, it must in fact be the case that either $\xi^{**} = \xi^*$ or $\xi^{**} = \xi^* + 1$. Now using Definition 1, in particular, equation (3), yields that since $y_1$ is chosen at $y_0$, we must have:

$$\xi^* \geq y_0 - \xi^{**} \tag{12}$$

  and $y_2$ is chosen at $y_1$ so:

$$y_1 - \xi^{**} \geq \xi^*$$

  which combine to give

$$y_1 \geq \xi^* + \xi^{**} \geq y_0,$$

  yielding a contradiction.

# Proof of Theorem III

- To prove the same result for median voter rules, the proof can be replicated with $y^*$ replaced by $\tilde{y}$. Instead of (12), we applied the definition in equation (4), which implies that as $y_1$ is a median voter choice at $y_0$, it must satisfy

$$\begin{array}{ll} y_0 \text{ odd:} & m(y_0) = (y_0+1)/2 \leq \xi^{**} \\ y_0 \text{ even:} & m(y_0) = y_0/2 \leq \xi^{*} \end{array} \tag{13}$$

  which exploits the fact that both $y_0/2$ and $(y_0/2)+1$ cannot be indifferent.

- Also as $y_2$ is the median voter choice at $y_1$, we have

$$\begin{array}{ll} y_1 \text{ odd:} & m(y_1) = (y_1+1)/2 > \xi^{*} \\ y_1 \text{ odd:} & m(y_1) = (y_1+1)/2 > \xi^{**} \end{array} \tag{14}$$

  again yielding a contradiction for all $\xi^{**} = \xi^*$ or $\xi^* + 1$, and completing the proof.

# Interpretation

- This theorem rules out extreme turning points in the size of club membership.
- An immediate implication is that there will not be cycles.

**Theorem**

*An MVE transition rule $y^*$ generates no cycles and a median voter rule $\tilde{y}$ generates no cycles.*

- Therefore, dynamic paths generated by $y^*$ and $\tilde{y}$ must, in a finite time, reach a steady state $\underline{x}$, $\underline{x} = y(\underline{x})$ where $y(\cdot)$ is $y^*(\cdot)$ or $\tilde{y}(\cdot)$.

# Monotonicity

- In fact, there is a stronger type of result:

**Theorem**

*If* $\{y_t\}$ *and* $\{y'_t\}$ *are dynamic paths generated by an MVE or a median voter rule then* $y_0 \geq y'_0 \Longrightarrow y_t \geq y'_t$ *for all* $t \geq 0$.

# Equivalence and Existence

**Theorem**

*An MVE transition rule* $y^*$ *is a median voter rule* $\tilde{y}$ *and vice versa.*

**Theorem**

*A (possibly stochastic) median voter rule* $\tilde{y}(\cdot)$ *exists.*

# Static Bliss Points

- Let $\mu(x)$ be the "*static (myopic) bliss point*" of this voter, i.e., the club size that would be optimal for a median voter who could commit to no further changes in the future:

$$\begin{aligned} x \text{ odd} &: \quad \mu(x) = \arg\max u\left(\cdot, \frac{x+1}{2}\right) & (15) \\ x \text{ even} &: \quad \mu(x) \in [\mu^*, \mu^{**}] & (16) \end{aligned}$$

where

$$\mu^* = \arg\max u\left(\cdot, \frac{x}{2}\right) \qquad \mu^{**} = \arg\max u\left(\cdot, \left(\frac{x}{2}\right) + 1\right).$$

# Steady States

- Consider an electorate of $x^*$ such that

$$u(x^*, m(x^*)) > u(x, m(x^*)) \qquad \text{for all } x, x \neq x^* \tag{17}$$

- Let $\tilde{y}(\cdot)$ be the median voter transition rule. Now, if $\tilde{y}(x^*) \neq x^*$ then

$$\begin{aligned} V(x^*, m(x^*), \tilde{y}) &= u(x^*, m(x^*)) + \delta V(\tilde{y}(x^*), m(x^*), \tilde{y}) \qquad (18) \\ &> V(\tilde{y}(x^*), m(x^*), \tilde{y}) \end{aligned}$$

  (Again weighted average reasoning).
- This implies:

**Theorem**

*If there is a club size $x^*$ where the unique value of $\mu(x^*)$ is $x^*$, then $x^*$ is a steady state of an MVE.*

- Intuitively, if a club size is reached which the median voter views as optimal then he will not wish to vote for a change in its size and, as he is a median voter, he can always enlist a majority in ensuring no change.

# Lack of Commitment

- Interestingly, there can be other types of steady states, that have the flavor of a *greater degree of inefficiency.*
- Consider a situation (with weak increasing differences) where preferences are as follows:

$$\begin{aligned} u(x, m(x^*)) &= -(x-(x^*+1))^2 \\ u(x, m(x^*+k)) &= -(x-(x^*+3))^2 \text{ for all } k \geq 1. \end{aligned} \tag{19}$$

- The future discounted utility of median voter $m(x^*)$:

$$\begin{array}{ll} \text{no change in club size:} & -\frac{1}{1-\delta} \\ \text{increase the club size:} & -\frac{\delta}{1-\delta}2^2 = -\frac{4\delta}{1-\delta} \end{array} \tag{20}$$

- Why? Therefore, MVE:

1. No change if $\delta \geq 1/4$
2. Increase to $x^*+1$ if $\delta < 1/4$

- If discounting is not too high, the club size of $x^*+4$ is a steady state even though the median voter (and all voters above the median) prefer a larger club.

# “Inefficient Steady States”

**Theorem**

*There can be MVE steady states involving clubs that are sub-optimal for the median voter at that club size.*

- Why? Current voting has to take into account the further changes being brought by tomorrow’s club.

# Voting Over Coalitions

- Another obvious example of dynamic voting with changing constituencies.
- Model based on Acemoglu, Egorov and Sonin (2007).
- A coalition, which will determine the distribution of a pie (more generally payoffs), both over its own membership.
- Possibility of future votes shaping the stability of current clubs illustrated more clearly.
- *Motivation:* the three-player divide the dollar game.

# Political Game

- Let $\mathcal{I}$ denote the collection of all individuals, which is assumed to be finite.
- The non-empty subsets of $\mathcal{I}$ are *coalitions* and the set of coalitions is denoted by $\mathcal{C}$.
- For any $X \subset \mathcal{I}$, $\mathcal{C}_X$ denotes the set of coalitions that are subsets of $X$ and $|X|$ is the number of members in $X$.
- In each period there is a designated *ruling coalition*, which can change over time.
- The game starts with ruling coalition $N$, and eventually the *ultimate ruling coalition* (URC) forms.
- When the URC is $X$, then player $i$ obtains *baseline* utility $w_i(X) \in \mathbb{R}$.
- $w(\cdot) \equiv \{w_i(\cdot)\}_{i \in \mathcal{I}}$.
- Important assumption: game of "non-transferable utility". Why?

# Political Power

- So far, our focus has been on "democratic" situations. One person one vote.
- Now allow differential powers across individuals.
- *Power* mapping to:

$$\gamma : \mathcal{I} \rightarrow \mathbb{R}_{++},$$

- $\gamma_i \equiv \gamma(i)$: political *power* of individual $i \in \mathcal{I}$ and $\gamma_X \equiv \sum_{i \in X} \gamma_i$ political power of coalition $X$.

# Winning Coalitions

- Coalition $Y \subset X$ is *winning* within coalition $X$ if and only if

  $$\gamma_Y > \alpha \gamma_X,$$

  where $\alpha \in [1/2, 1)$ is a (weighted) supermajority rule ($\alpha = 1/2$ corresponds to majority rule).
- Let us write: $Y \in \mathcal{W}_X$ for $Y \subset X$ winning within $X$.
- Since $\alpha \geq 1/2$, if $Y, Z \in \mathcal{W}_X$, then $Y \cap Z \neq \varnothing$.

# Payoffs

**Assumption:** Let $i \in I$ and $X, Y \in C$. Then:
(1) If $i \in X$ and $i \notin Y$, then $w_i(X) > w_i(Y)$ [i.e., each player prefers to be part of the URC].
(2) For $i \in X$ and $i \in Y$ , $w_i(X) > w_i(Y) \iff \gamma_i/\gamma_X > \gamma_i/\gamma_Y$ ($\iff \gamma_X < \gamma_Y$) [i.e., for any two URCs that he is part of, each player prefers the one where his relative power is greater].
(3) If $i \notin X$ and $i \notin Y$, then $w_i(X) = w_i(Y) \equiv w_i^-$ [i.e., a player is indifferent between URCs he is not part of].

- Interpretation.
- Example:

$$w_i(X) = \frac{\gamma_{X \cap \{i\}}}{\gamma_X} = \begin{cases} \gamma_i/\gamma_X & \text{if } i \in X \\ 0 & \text{if } i \notin X \end{cases} . \tag{21}$$

# Extensive-Form Game

- Choose $\varepsilon > 0$ arbitrarily small. Then, the extensive form of the game $\Gamma = \left(N, \gamma|_N, w\left(\cdot\right), \alpha\right)$ is as follows. Each *stage j* of the game starts with some ruling coalition $N_j$ (at the beginning of the game $N_0 = N$). Then:

1. Nature randomly picks agenda setter $a_{j,q} \in N_j$ for $q = 1$.
2. [Agenda-setting step] Agenda setter $a_{j,q}$ makes proposal $P_{j,q} \in \mathcal{C}_{N_j}$, which is a subcoalition of $N_j$ such that $a_{j,q} \in P_{j,q}$ (for simplicity, we assume that a player cannot propose to eliminate himself).
3. [Voting step] Players in $P_{j,q}$ vote sequentially over the proposal. More specifically, Nature randomly chooses the first voter, $v_{j,q,1}$, who then casts his vote vote $\tilde{v}\left(v_{j,q,1}\right) \in \{\tilde{y}, \tilde{n}\}$ (Yes or No), then Nature chooses the second voter $v_{j,q,2} \neq v_{j,q,1}$ etc. After all $\left|P_{j,q}\right|$ players have voted, the game proceeds to step 4 if players who supported the proposal form a winning coalition within $N_j$ (i.e., if $\left\{i \in P_{j,q} : \tilde{v}\left(i\right) = \tilde{y}\right\} \in \mathcal{W}_{N_j}$), and otherwise it proceeds to step 5.

# Extensive-Form Game (continued)

4. If $P_{j,q} = N_j$, then the game proceeds to step 6. Otherwise, players from $N_j \setminus P_{j,q}$ are eliminated and the game proceeds to step 1 with $N_{j+1} = P_{j,q}$ (and $j$ increases by 1 as a new transition has taken place).
5. If $q < |N_j|$, then next agenda setter $a_{j,q+1} \in N_j$ is randomly picked by Nature among members of $N_j$ who have not yet proposed at this stage (so $a_{j,q+1} \neq a_{j,r}$ for $1 \leq r \leq q$), and the game proceeds to step 2 (with $q$ increased by 1). If $q = |N_j|$, the game proceeds to step 6.
6. $N_j$ becomes the ultimate ruling coalition. Each player $i \in N$ receives total payoff

$$U_i = w_i \left( N_j \right) - \varepsilon \sum\nolimits_{1 \leq k \leq j} \mathbf{I}_{\{i \in N_k\}}, \tag{22}$$

where $\mathbf{I}_{\{\cdot\}}$ is the indicator function taking the value of 0 or 1.

# Discussion

- Natural game of sequential choice of coalitions.
- $\varepsilon$ introduced for technical reasons (otherwise, indifferences lead to uninteresting transitions).
- Important assumption: players eliminated have no say in the future.
- Stark representation of changing constituencies, but not a good approximation to democratic decision-making.
- More reminiscent to "dealmaking in autocracies"—or coalition formation in nondemocracies.

# Axiomatic Analysis

- Games of coalition formation have noncooperative and cooperative features.
- Ideally, the two perspectives give congruent results.
- Key idea in the extensive-form game: players will not support a coalition that will later eliminate themselves.
  →*stability*
- Let us first capture this notion using an axiomatic approach.

# Axioms

- Consider the set of games $(N, \gamma|_N, w(\cdot), \alpha)$.
- Holding $\gamma$, $w$ and $\alpha$ fixed, consider the correspondence $\phi : \mathcal{C} \rightrightarrows \mathcal{C}$ defined by $\phi(N) = \Phi(N, \gamma|_N, w, \alpha)$ for any $N \in \mathcal{C}$.
- Axioms on $\phi$.

**Axiom 1** (Inclusion) For any $X \in \mathcal{C}$, $\phi(X) \neq \varnothing$ and if $Y \in \phi(X)$, then $Y \subset X$.
**Axiom 2** (Power) For any $X \in \mathcal{C}$, $Y \in \phi(X)$ only if $Y \in \mathcal{W}_X$.
**Axiom 3** (Self-Enforcement) For any $X \in \mathcal{C}$, $Y \in \phi(X)$ only if $Y \in \phi(Y)$.
**Axiom 4** (Rationality) For any $X \in \mathcal{C}$, for any $Y \in \phi(X)$ and for any $Z \subset X$ such that $Z \in \mathcal{W}_X$ and $Z \in \phi(Z)$, we have that $Z \notin \phi(X) \iff \gamma_Y < \gamma_Z$.

- Interpretation.

# Self-Enforcing Coalitions

- Motivated by the self-enforcement axiom:

**Definition**

*Coalition* $X \in P(\mathcal{I})$ *is* self-enforcing *if* $X \in \phi(X)$.

**Assumption:** The power mapping $\gamma$ is *generic* in the sense that if for any $X, Y \in \mathcal{C}$, $\gamma_X = \gamma_Y$ implies $X = Y$.

- We also say that coalition $N$ is generic or that numbers $\{\gamma_i\}_{i \in N}$ are generic if mapping $\gamma|_N$ is generic.

# Main Axiomatic Result

## Theorem

*Fix* $\mathcal{I}$, $\gamma$, $w(\cdot)$ *and* $\alpha \in [1/2, 1)$. *Then:*
*1. There exists a unique mapping* $\phi$ *that satisfies Axioms* 1–4. *Moreover, when* $\gamma$ *is generic,* $\phi$ *is single-valued.*
*2. This mapping* $\phi$ *may be obtained by the following inductive procedure. For any* $k \in \mathbb{N}$, *let* $\mathcal{C}^k = \{X \in \mathcal{C} : |X| = k\}$. *Clearly,* $\mathcal{C} = \cup_{k \in \mathbb{N}} \mathcal{C}^k$. *If* $X \in \mathcal{C}^1$, *then let* $\phi(X) = \{X\}$. *If* $\phi(Z)$ *has been defined for all* $Z \in \mathcal{C}^n$ *for all* $n < k$, *then define* $\phi(X)$ *for* $X \in \mathcal{C}^k$ *as*

$$\phi(X) = \underset{A \in \mathcal{M}(X) \cup \{X\}}{\operatorname{argmin}} \gamma_A, \tag{23}$$

*where*

$$\mathcal{M}(X) = \{Z \in \mathcal{C}_X \setminus \{X\} : Z \in \mathcal{W}_X \text{ and } Z \in \phi(Z)\}. \tag{24}$$

*Proceeding inductively* $\phi(X)$ *is defined for all* $X \in \mathcal{C}$.

# Intuition

- For each $X$, (24) defines $\mathcal{M}(X)$ as the set of proper subcoalitions which are both winning and self-enforcing. Equation (23) then picks the coalitions in $\mathcal{M}(X)$ that have the least power.
- When there are no proper winning and self-enforcing subcoalitions, $\mathcal{M}(X)$ is empty and $X$ becomes the URC), which is captured by (23).
- What does this mean?

# Implication

**Corollary**

*Coalition $N$ is self-enforcing, that is, $N \in \phi(N)$, if and only if there exists no coalition $X \subset N$, $X \neq N$, that is winning within $N$ and self-enforcing. Moreover, if $N$ is self-enforcing, then $\phi(N) = \{N\}$.*

- Main implication: a coalition that includes a winning and self-enforcing subcoalition cannot be self-enforcing. This captures the notion that the stability of smaller coalitions undermines stability of larger ones.
- Application: coalition formation among three players with approximately equal powers.

# Noncooperative Game

**Theorem**

*Suppose that* $\phi(N)$ *satisfies Axioms 1-4 (cfr.* $(23)$ *in the axiomatic analysis). Then, for any* $K \in \phi(N)$*, there exists a pure strategy profile* $\sigma_K$ *that is an SPE and leads to URC* $K$ *in at most one transition. In this equilibrium player* $i \in N$ *receives payoff*

$$U_i = w_i(K) - \varepsilon \mathbf{I}_{\{i \in K\}} \mathbf{I}_{\{N \neq K\}}. \tag{25}$$

*This equilibrium payoff does not depend on the random moves by Nature.*

- Thus equivalence between cooperative and noncooperative approaches.

# Intuition

- Suppose each player anticipates members of a self-enforcing ruling coalition to play a strategy profile such that they will turn down any offers other than $K$ and they will accept $K$;
- then, it is in the interest of all the players in $K$ to play such a strategy for any history.
- This follows immediately because by the definition of the set $\phi(N)$, because for any deviation to be profitable, the URC that emerges after such deviation must be either not self-enforcing or not winning.
- But the the first option is ruled out by induction while a deviation to a non-winning URC will be blocked by sufficiently many players.
- The payoff in (25) is also intuitive.
- Each player receives his baseline payoff $w_i(K)$ resulting from URC $K$ and then incurs the cost $\varepsilon$ if he is part of $K$ and if the initial coalition $N$ is not equal to $K$ (because in this latter case, there will be one transition).

# Stronger Results Under Genericity

**Theorem**

*Suppose the genericity Assumption holds and suppose* $\phi(N) = K$. *Then any (pure or mixed strategy) SPE results in* $K$ *as the URC. The payoff player* $i \in N$ *receives in this equilibrium is given by* (25).

- Intuition.

# Characterization

- Equilibrium characterize simply by a set of recursive equations.
- What are the implications of equilibrium coalition formation
- Let us impose one more assumption

**Assumption:** *For no* $X, Y \in \mathcal{C}$ *such that* $X \subset Y$ *the equality* $\gamma_Y = \alpha\gamma_X$ *is satisfied.*

# Continuity of Ruling Coalitions

**Proposition** Consider $\Gamma = (N, \gamma, w(\cdot), \alpha)$ with $\alpha \in [1/2, 1)$. Then:
1. There exists $\delta > 0$ such that if $\gamma' : N \to \mathbb{R}_{++}$ lies within $\delta$-neighborhood of $\gamma$, then $\Phi(N, \gamma, w, \alpha) = \Phi(N, \gamma', w, \alpha)$.
2. There exists $\delta' > 0$ such that if $\alpha' \in [1/2, 1)$ satisfies $|\alpha' - \alpha| < \delta'$, then $\Phi(N, \gamma, w, \alpha) = \Phi(N, \gamma, w, \alpha')$.
3. Let $N = N_1 \cup N_2$ with $N_1$ and $N_2$ disjoint. Then, there exists $\delta > 0$ such that for all $N_2$ such that $\gamma_{N_2} < \delta$, $\phi(N_1) = \phi(N_1 \cup N_2)$.

# Fragility of Self-Enforcing Coalitions

**Proposition** Suppose $\alpha = 1/2$ and fix a power mapping $\gamma : \mathcal{I} \rightarrow \mathbb{R}_{++}$. Then:
1. If coalitions $X$ and $Y$ such that $X \cap Y = \varnothing$ are both self-enforcing, then coalition $X \cup Y$ is not self-enforcing.
2. If $X$ is a self-enforcing coalition, then $X \cup \{i\}$ for $i \notin X$ and $X \setminus \{i\}$ for $i \in X$ are not self-enforcing.

- Implication: under majority rule $\alpha = 1/2$, the addition or the elimination of a single agent from a self-enforcing coalitions makes this coalition no longer self-enforcing. Why?

# Size of Ruling Coalitions I

**Proposition** Consider $\Gamma = (N, \gamma, w(\cdot), \alpha)$.
1. Suppose $\alpha = 1/2$, then for any $n$ and $m$ such that $1 \leq m \leq n$, $m \neq 2$, there exists a set of players $N$, $|N| = n$, and a generic mapping of powers $\gamma$ such that $|\phi(N)| = m$. In particular, for any $m \neq 2$ there exists a self-enforcing ruling coalition of size $m$. However, there is no self-enforcing coalition of size 2.
2. Suppose that $\alpha > 1/2$, then for any $n$ and $m$ such that $1 \leq m \leq n$,there exists a set of players $N$, $|N| = n$, and a generic mapping of powers $\gamma$ such that $|\phi(N)| = m$.

- Therefore, one can say relatively little about the size and composition of URCs without specifying the power distribution within the society further (except that when $\alpha = 1/2$, coalitions of size 2 are not self-enforcing).
- But this is because no discipline on the distribution of powers.

# Size of Ruling Coalitions II

**Proposition** Consider $\Gamma = (N, \gamma, w(\cdot), \alpha)$ with $\alpha \in [1/2, 1)$. Suppose that there exists $\delta > 0$ such that $\max_{i,j \in N} \{\gamma_i / \gamma_j\} < 1 + \delta$. Then:
1. When $\alpha = 1/2$, any ruling coalition must have size $k_m = 2^m - 1$ for some $m \in \mathbb{Z}$, and moreover, $\phi(N) = N$ if and only if $|N| = k_m$ for $k_m = 2^m - 1$.
2. When $\alpha \in [1/2, 1)$, $\phi(N) = N$ if and only if $|N| = k_{m,\alpha}$ where $k_{1,\alpha} = 1$ and $k_{m,\alpha} = \lfloor k_{m-1,\alpha}/\alpha \rfloor + 1$ for $m > 1$, where $\lfloor z \rfloor$ denotes the integer part of $z$.

- When powers are approximately equal, the size of the URC is determined tightly.

# Rules and Coalitions

- Should an increase in $\alpha$ raise the size of the URC? Should an individual always gain from an increase in his power?
- Intuitive, but the answers are no and no.

**Proposition**

1. An increase in $\alpha$ may reduce the size of the ruling coalition. That is, there exists a society $N$, a power mapping $\gamma$ and $\alpha, \alpha' \in [1/2, 1)$, such that $\alpha' > \alpha$ but for all $X \in \Phi(N, \gamma, w, \alpha)$ and $X' \in \Phi(N, \gamma, w, \alpha')$, $|X| > |X'|$ and $\gamma_X > \gamma_{X'}$.
2. There exist a society $N$, $\alpha \in [1/2, 1)$, two mappings $\gamma, \gamma' : N \to \mathbb{R}_{++}$ satisfying $\gamma_i = \gamma'_i$ for all $i \neq j$, $\gamma_j < \gamma'_j$ such that $j \in \Phi(N, \gamma, w, \alpha)$, but $j \notin \Phi(N, \gamma', w, \alpha)$. Moreover, this result applies even when $j$ is the most powerful player in both cases, i.e. $\gamma'_i = \gamma_i < \gamma_j < \gamma'_j$ for all $i \neq j$.

- Why?

# Power and Ruling Coalitions

- When will the most powerful individual be part of the ruling coalition?

**Proposition** Consider the game $\Gamma\left(N, \gamma, w\left(\cdot\right), \alpha\right)$ with $\alpha \in [1/2, 1)$, and suppose that $\gamma_1, \ldots, \gamma_{|N|}$ is an increasing sequence. If $\gamma_{|N|} \in \left(\alpha \sum_{j=2}^{|N|-1} \gamma_j / \left(1-\alpha\right), \alpha \sum_{j=1}^{|N|-1} \gamma_j / \left(1-\alpha\right)\right)$, then either coalition $N$ is self-enforcing or the most powerful individual, $|N|$, is not a part of the URC.

- Intuition?

# Conclusions

- Once dynamic voting also affects the distribution of political power, richer set of issues arise.
- Endogeneity of constituencies is both practically relevant and related to endogenous institutions.
- Ensuring equilibria in situations of dynamic voting harder, but often we can put economically interesting structure to ensure equilibria (once we know what we are trying to model).