14.451: Introduction to Economic Growth
Problem Set 3

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Question 1: Explain why the Arrow sufficiency theorem cannot be applied to conclude that the representative household optimization problem in the neoclassical growth model has a unique solution. Construct an argument exploiting the fact that the representative household's Hamiltonian in the neoclassical growth model, $H(t, a, c, \mu)$, is concave in (a, c) and strictly concave in c to show that the path of consumption and assets that satisfies the necessary conditions implied by the Maximum Principle and the strong transversality condition constitutes the unique solution to the household's maximization problem.

Question 2: Consider the baseline neoclassical model with no technological progress.

1. Show that in the neighborhood of the steady state k^*, the law of motion of $k(t) \equiv K(t)/L(t)$ can be represented as

$$k(t) = k^* + \eta_1 \exp(\xi_1 t) + \eta_2 \exp(\xi_2 t),$$

   where ξ_1 and ξ_2 are the eigenvalues of the linearized system.

2. Compute these eigenvalues show that one of them, say ξ_2, is positive.

3. What does this imply about the value of η_2?

4. How is the value of η_1 determined? (For this part, assume that the equation in part 1 holds exactly).

5. What determines the speed of adjustment of $k(t)$ towards its steady-state value k^*?

Question 3: Consider the basic neoclassical growth model with CRRA preferences, but with consumer heterogeneity in initial asset holdings (you may assume no technological change if you wish). In particular, there is a set $\mathcal{H}$ of household and household $h \in \mathcal{H}$ starts with initial assets $a_h(0)$. Households are otherwise identical.

1. Characterize the competitive equilibrium of this economy and show that the behavior of per capita variables is identical to that in a representative household economy, with the representative household starting with assets $a(0) = |\mathcal{H}|^{-1} \int_{\mathcal{H}} a_h(0)\, dh$, where $|\mathcal{H}|$ is the measure (number) of households in this economy. Interpret this result and relate it to the Aggregation Theorem.

2. Show that if, instead of the no-Ponzi condition, we impose $a_h(t) \geq 0$ for all $h \in \mathcal{H}$ and for all t, then a different equilibrium allocation may result. In light of this finding, discuss whether (and when) it is appropriate to use a no-borrowing constraint instead of the no-Ponzi condition.

Question 4: Consider the standard neoclassical growth model augmented with labor supply decisions. In particular, there is a total population normalized to 1, and all individuals have utility function

$$U(0) = \int_0^{\infty} \exp(-\rho t)\, u(c(t), 1 - l(t)),$$

where $l(t) \in (0,1)$ is labor supply. In a symmetric equilibrium, employment $L(t)$ is equal to $l(t)$. Assume that the production function is given by $Y(t) = F[K(t), A(t) L(t)]$, which satisfies all the standard assumptions and $A(t) = \exp(gt) A(0)$.

1. Define a competitive equilibrium.

2. Set up the current-value Hamiltonian that each household solves taking wages and interest rates as given, and determine the necessary and sufficient conditions for the allocation of consumption over time and leisure-labor trade off.

3. Set up the current-value Hamiltonian for a planner maximizing the utility of the representative household, and derive the necessary and sufficient conditions for an optimal solution.

4. Show that the two problems are equivalent given competitive markets.

5. Show that unless the utility function is asymptotically equal to

$$u(c(t), 1 - l(t)) = \begin{cases} \frac{Ac(t)^{1-\theta}}{1-\theta} h(1 - l(t)) & \text{for } \theta \neq 1, \\ A \log c(t) + Bh(1 - l(t)) & \text{for } \theta = 1, \end{cases}$$

for some $h(\cdot)$ with $h'(\cdot) > 0$, there will not exist a BGP with constant and equal rates of consumption and output growth, and a constant level of labor supply. Explain why this is the only functional form consistent with BGP.

Question 5: Consider the two-period canonical overlapping generations model with log preferences

$$\log\left(c_1\left(t\right)\right)+\beta\log\left(c_2\left(t+1\right)\right)$$

for each individual. Suppose that there is population growth at the rate n. Individuals work only when they are young, and supply one unit of labor inelastically. Production technology is given by

$$Y\left(t\right)=A\left(t\right)K\left(t\right)^{\alpha}L\left(t\right)^{1-\alpha},$$

where $A\left(t+1\right)=\left(1+g\right)A\left(t\right)$, with $A\left(0\right)>0$ and $g>0$.

1. Define a competitive equilibrium and the steady-state equilibrium.
2. Can you apply the First Welfare Theorem to this competitive equilibrium? Carefully explain your answer.
3. Can you apply the Second Welfare Theorem? Be specific about how you would decentralize the Pareto optimal allocations?
4. Characterize the steady-state equilibrium and show that it is globally asymptotically stable.
5. What is the effect of an increase in g on the equilibrium path?

Question 6: Consider the baseline overlapping generations model and suppose that the equilibrium is dynamically efficient, i.e., $r^{*}>n$. Show that any unfunded Social Security system will increase the welfare of the current old generation and reduce the welfare of some future generation.