Supplementary Material for "Bubbles and Self-Enforcing Debt"

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In this document, we present three extensions of the example in Section 3. First, we discuss how initial debt limits and initial asset holdings determine the transition to a steady-state equilibrium. Second, we show the existence of non-stationary equilibria, in which the real value of debt collapses over time. These equilibria are the counter-part to the "hyper-inflation" equilibria that exist in the environment with unbacked public debt. Finally, we illustrate how our characterization of self-enforcing debt extends to environments with growth, showing that, with CRRA utility, a stationary equilibrium of our model is characterized by a real interest rate that is equal to the aggregate growth rate.

1 Transitional dynamics

In the environment with unbacked public debt, it is well known that the transition to steady-state is complete the first time the state switches. Before then, the consumption allocations of each type are determined by initial asset holdings. Here, we show that the same result applies to the economy with self-enforcing private debt, except that consumption allocations in the initial phase are determined by both the debt limits and the initial asset holdings of each type.

We begin by showing that the steady-state allocation $(\overline{c}, \underline{c})$ of Proposition 1 in the main text does not require debt limits to be identical for both types - instead, the same allocations and prices are sustained by any debt limits $\left(-\omega^1, -\omega^2\right)$, such that $\omega^1 + \omega^2 = 2\omega$. To see that the consumption allocations $(\overline{c}, \underline{c})$ and steady-state state prices (q_c, q_{nc}) continue to characterize an equilibrium with self-enforcing debt even when debt limits are asymmetric, consider asset holdings of $a^j\left(s^t\right) = -\omega^j$, if $s_t = \mathsf{s}_j$, and $a^j\left(s^t\right) = \omega^{-j}$, if $s_t \neq \mathsf{s}_j$. These asset holdings clear the market, and yield $c^j\left(s^t\right) =$

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$\overline{e} - \omega^j (1 - q_{nc}) - \omega^{-j} q_c = \overline{e} - q_c \left(\omega^j + \omega^{-j}\right) = \overline{c}$ if $s_t = \mathsf{s}_j$, and $c^j \left(s^t\right) = \underline{e} + \omega^{-j} (1 - q_{nc}) + \omega^j q_c = \underline{c}$, if $s_t \neq \mathsf{s}_j$. Therefore, the steady-state allocations can be supported by a continuum of different debt limits. In the extreme case, where $\omega^1 = 0$, type 1 never borrows, and type 2 never lends.

Now, suppose that the economy begins at date 0 in state $s_0 = \mathsf{s}_1$, and initial asset holdings are $a^1 (s_0) = a = -a^2 (s_0)$, for some value of a, and steady-state debt limits are $\left(-\omega^1, -\omega^2\right)$, with $\omega^1 + \omega^2 = 2\omega$. We construct equilibria where the consumption allocation is constant and equal to $(\overline{c}^o, \underline{c}^o)$, asset holdings are $a^1 \left(s^t\right) = a = -a^2 \left(s^t\right)$ as long as $s_t = \mathsf{s}_1$, and it switches to the steady state allocation $(\overline{c}, \underline{c})$ the first time $s_t = \mathsf{s}_2$. Let (q_c^o, q_{nc}^o) and $\left(-\omega^{1o}, -\omega^{2o}\right)$, and denote, respectively, the state-contingent prices and the debt limits of types 1 and 2 in the transitional phase (also assumed to be constant during this phase). $(\overline{c}^o, \underline{c}^o)$ and (q_c^o, q_{nc}^o) satisfy the consumer's budget constraints and first-order conditions:

$$\overline{c}^o = \overline{e} + a (1 - q_{nc}^o) - q_c^o \omega^2 \text{ and } \underline{c}^o = \underline{e} - a (1 - q_{nc}^o) + q_c^o \omega^2 \qquad (1)$$

$$q_c^o = \beta\alpha \frac{u'(\underline{c})}{u'(\overline{c}^o)} \text{ and } q_{nc}^o = \beta (1 - \alpha). \qquad (2)$$

The initial debt limits $\left(\omega^{1o}, \omega^{2o}\right)$ then satisfy the exact roll-over condition, which requires that $\omega^{jo} = q_{nc}^o \omega^{jo} + q_c^o \omega^j$, or

$$\omega^{jo} = \frac{\beta\alpha}{1 - \beta (1 - \alpha)} \frac{u'(\underline{c})}{u'(\overline{c}^o)} \omega^j. \qquad (3)$$

Substituting the condition for $\overline{c}^o$ into the one for q_c^o and rearranging, we find

$$\overline{c}^o + \beta\alpha \frac{u'(\underline{c})}{u'(\overline{c}^o)} \omega^2 = \overline{e} + a (1 - \beta (1 - \alpha)) \qquad (4)$$

Since the LHS of (4) is strictly increasing in $\overline{c}^o$, (4) admits a unique solution, from which one can solve for the other variables. We thus have the following characterization of equilibrium transition paths.

Proposition 1 *For given initial asset holdings $a^1 (s_0) = a = -a^2 (s_0)$ and steady-state debt limits $\left(-\omega^1, -\omega^2\right)$, consider state-contingent prices, debt limits and consumption allocations (q_c^o, q_{nc}^o), $\left(-\omega^{1o}, -\omega^{2o}\right)$, and $(\overline{c}^o, \underline{c}^o)$ for the transitional phase prior to the first time the state switches from s_1 to s_2. This characterizes a competitive equilibrium, if and only if (1), (2), and (3) are satisfied, and initial asset holdings satisfy $a \in \left[-\omega^{1o}, \omega^{2o}\right]$.*

Proof. Conditions (1), (2), and (3), together with $a \in \left[-\omega^{1o}, \omega^{2o}\right]$ and $\overline{c}^o \geq \underline{c}$ are necessary and sufficient conditions for the characterization of a competitive equilibrium of the form that we construct here; the requirement that $a \in \left[-\omega^{1o}, \omega^{2o}\right]$ implies that debt limits for both types are satisfied during the transition phase, while $\overline{c}^o \geq \underline{c}$ implies that the type 2 agent's first-order condition holds as an inequality, when the state changes.

To prove our result, we thus need to show that this last condition is redundant, i.e. that it is always implied by the former. Using (4), one finds that $\overline{c}^o$ is an increasing function of initial asset holdings a. Moreover, rearranging (4) in terms of q_c^o, we have $q_c^o u'\left(\overline{e} + a\left(1 - q_{nc}^o\right) - q_c^o \omega^2\right) = \beta\alpha u'\left(\underline{c}\right)$. When $a = -\omega^{1o}$, the LHS of this expression reduces to $q_c^o u'\left(\overline{e} - q_c^o\left(\omega^1 + \omega^2\right)\right) = \beta\alpha u'\left(\underline{c}\right)$, from which it follows that $q_c^o = q_c$ and $\overline{c}^o = \overline{c}$ when $a = -\omega^{1o}$. It follows that for any $a \geq -\omega^{1o}$, $\overline{c}^o \geq \overline{c} > \underline{c}$, so that the type 2 agent's first-order condition is satisfied. Finally, notice that when $a = \omega^{2o}$, $\overline{c}^o = \overline{e}$ - for any higher initial asset position of type 1 (and lower asset position of type 2), type 2 would have a strict incentive to default, consume his autarky allocation for one period, and then reenter the market purely as a lender. ■

Thus, the amount of consumption smoothing that is feasible during the transition phase is a function of the debt limits allocated to each type, and the initial asset positions. In the special case where type 1's initial asset position is exactly at his debt limit, the economy starts out directly in the steady-state equilibrium. On the other hand, if type 2's initial asset position is at his debt limit, only the autarky allocation is feasible during the transition phase, and risk sharing starts only once there is a switch in states. For any intermediate configuration, the extent of consumption-smoothing in the transition depends on how far each type is from his debt limit - the further type 2 is from his limit, and the closer type 1 is to his, the more consumption smoothing is feasible.

2 Non-stationary equilibria

We begin by considering non-stationary equilibrium paths in the example of Section 3 of the paper. Let $\mathcal{K}\left(s^t\right)$ denote the number of times the state has switched from s_1 to s_2 or from s_2 to s_1 along history s^t. We construct equilibria that are characterized by a sequence $\left\{q_{nc}^k, q_c^{k+1}, \overline{c}^k, \underline{c}^k, \omega^k\right\}_{k=0}^{\infty}$, where asset prices are $q\left(s^{t+1}\right) = q_c^k$ if $s_{t+1} \neq s_t$ and $k = \mathcal{K}\left(s^t\right)$ and $q\left(s^{t+1}\right) = q_{nc}^k$ if $s_{t+1} = s_t$ and $k = \mathcal{K}\left(s^t\right)$, consumption allocations are $c^j\left(s^t\right) = \overline{c}^k$ if $s_t = \mathsf{s}_j$ and $k = \mathcal{K}\left(s^t\right)$ and $c^j\left(s^t\right) = \underline{c}^k$ if $s_t \neq \mathsf{s}_j$ and $k = \mathcal{K}\left(s^t\right)$, asset holdings are $a^j\left(s^t\right) = -\omega^k$ if $s_t = \mathsf{s}_j$ and $k = \mathcal{K}\left(s^t\right)$ and $a^j\left(s^t\right) = \omega^k$ if $s_t \neq \mathsf{s}_j$ and $k = \mathcal{K}\left(s^t\right)$, and debt limits are $\phi^j\left(s^t\right) = \omega^k$ if $k = \mathcal{K}\left(s^t\right)$. That is, as in the stationary equilibrium, agents are constrained at low-endowment histories, but the tightness of the constraint changes each time the state switches between s_1 and s_2.

To construct the sequence $\left\{q_{nc}^k, q_c^{k+1}, \overline{c}^k, \underline{c}^k, \omega^k\right\}_{k=0}^{\infty}$, notice that the consumption allocations and prices must satisfy the agents' budget constraint and first-order conditions at high-endowment histories:

$$\overline{c}^k = \overline{e} - \omega^k + q_{nc}^k \omega^k - q_c^{k+1}\omega^{k+1} \tag{5}$$

$$\underline{c}^k = \underline{e} + \omega^k - q_{nc}^k \omega^k + q_c^{k+1}\omega^{k+1} \tag{6}$$

$$q_c^{k+1} u'\left(\overline{c}^k\right) = \beta\alpha u'\left(\underline{c}^{k+1}\right) \tag{7}$$

$$q_{nc}^k = \beta(1-\alpha) \tag{8}$$

In addition, the sequence of debt limits must satisfy the exact roll-over condition:

$$\omega^k = q_c^{k+1}\omega^{k+1} + q_{nc}^k \omega^k. \tag{9}$$

Substituting (5)-(8) into (9), and then using (9), the dynamics of ω^k are then characterized by the following difference equation:

$$\omega^{k+1}\beta\alpha u'\left(\underline{e} + 2(1-\beta(1-\alpha))\omega^{k+1}\right) - (1-\beta(1-\alpha))\omega^k u'\left(\overline{e} - 2(1-\beta(1-\alpha))\omega^k\right) = 0 \tag{10}$$

This difference equation has two stationary points at ω (the steady state value derived in Proposition 1 in the paper), and the other at zero. Moreover, we can rearrange this difference equation in the form $\omega^k = F\left(\omega^{k+1}\right)$, where the function F is continuous and has the property that if $\omega^{k+1} > \omega$, then $F\left(\omega^{k+1}\right) > \omega^{k+1}$, and if $\omega^{k+1} < \omega$, then $F\left(\omega^{k+1}\right) < \omega^{k+1}$. This in turn implies that for each $\omega^k < \omega$, there exists $\omega^{k+1} < \omega^k$ for which $\omega^k = F\left(\omega^{k+1}\right)$.[1] We thus have the following characterization of non-stationary equilibria:

Proposition 2 *For given $\omega^0 \in (0, \omega)$, there exists a decreasing sequence $\left\{\omega^k\right\}_{k=0}^{\infty}$ that is recursively defined by (10), and a non-stationary equilibrium of the economy in Section 3, where prices and allocations are given by (5)-(8), for $k = 0, 1, 2, ...$*

Proof. To complete the above argument, we just need to check the agents' first-order conditions for low-endowment periods, which require $q_c^{k+1} u'\left(\underline{c}^k\right) \geq \beta\alpha u'\left(\overline{c}^{k+1}\right)$, or equivalently $u'\left(\overline{c}^k\right)/u'\left(\underline{c}^k\right) \leq u'\left(\underline{c}^{k+1}\right)/u'\left(\overline{c}^{k+1}\right)$. Using the fact that $\omega^k < \omega$ for all k, we have $\overline{c}^k > \overline{c}$ and $\underline{c}^k < \underline{c}^k$ for all k, and therefore $u'\left(\overline{c}^k\right)/u'\left(\underline{c}^k\right) \leq u'(\overline{c})/u'(\underline{c}) < 1$ and $u'\left(\underline{c}^{k+1}\right)/u'\left(\overline{c}^{k+1}\right) \geq u'(\underline{c})/u'(\overline{c}) > 1$, from which the result follows immediately. ■

These non-stationary equilibria are characterized by a self-fulfilling collapse of the value of debt: agents anticipate that debt limits will tighten in the future, which limits the incentives for

[1] If F is invertible, then this is the unique equilibrium path starting from any equilibrium value of $\omega^0 \leq \omega$. If F is not invertible, there may be other solutions to (10), some of which satisfy $\omega^{k+1} \geq \omega^k$. A sufficient condition for invertibility is $-u''(c)\,c/u'(c) \leq 1$, for $c \in (0,1)$.

repayment, and hence tightens current debt limits. These equilibria correspond to the 'hyper-inflationary' equilibria of the economy with unbacked public debt, in which the real value of public debt gradually collapses.

3 Growing endowments

Consider a variation on the economy of Section 3, where the two types still receive randomly alternating endowments but the aggregate endowments are stochastically growing over time. Uncertainty is represented by the Markov process $h_t = s_t \times z_t \in S \times Z$, where $S = \{\mathsf{s}_1, \mathsf{s}_2\}$ determines the share of aggregate endowments going to each type, and $Z = \{z_1, ..., z_N\}$ determines the growth rate of aggregate endowments. Endowments $y^j\left(h^t\right)$ are thus given by

$$\begin{aligned} y^j\left(h^t\right) &= \overline{e} f\left(z^t\right) \text{ if } s_t = \mathsf{s}_j, \\ y^j\left(h^t\right) &= \underline{e} f\left(z^t\right) \text{ if } s_t \neq \mathsf{s}_j, \end{aligned}$$

with $\overline{e} + \underline{e} = 1$, and aggregate endowments characterized recursively by $f\left(z^t\right) = g\left(z_t\right) \cdot f\left(z^{t-1}\right)$. Transition probabilities are defined by $\pi\left(h_{t+1} | h_t\right) = \Pr\left[s_{t+1} | s_t\right] \cdot \Pr\left[z_t\right]$; that is, aggregate and distributional shocks are independent of each other, and the growth rate of aggregate endowments is i.i.d. over time. The distributional shocks are characterized as before by symmetric transition probabilities $\Pr\left[s_{t+1} = \mathsf{s}_1 | s_t = \mathsf{s}_2\right] = \Pr\left[s_{t+1} = \mathsf{s}_2 | s_t = \mathsf{s}_1\right] = \alpha$. Agents have CRRA utility, $u\left(c\right) = c^{1-\sigma} / \left(1-\sigma\right)$. We further assume that $\beta \sum_{z'} \Pr\left[z'\right] g\left(z'\right)^{1-\sigma} < 1$, so that life-time expected utilities are finite.

We solve this extension of our model for a stationary equilibrium, in which state-prices, and consumption allocations, asset holdings and debt limits (normalized by aggregate endowments) are functions only of the current state h_t. Following the same steps as Alvarez and Jermann, we can re-cast this extension as an economy with constant endowments. In particular, consider an economy with aggregate endowments normalized to 1 for all z^t, the probability of state z' given by $\hat{\pi}\left(z'\right) = \Pr\left[z'\right] g\left(z'\right)^{1-\sigma} / \sum_{z'} \Pr\left[z'\right] g\left(z'\right)^{1-\sigma}$, and discount rate given by $\hat{\beta} = \beta \sum_{z'} \Pr\left[z'\right] g\left(z'\right)^{1-\sigma}$.[2] Consider $\left\{\hat{C}, \hat{Y}, \hat{A}, \hat{\Phi}, \hat{Q}\right\}$ such that

$$\hat{c}^j\left(h^t\right) = \frac{c^j\left(h^t\right)}{f\left(z^t\right)}, \quad \hat{y}^j\left(h^t\right) = \frac{y^j\left(h^t\right)}{f\left(z^t\right)}, \quad \hat{a}^j\left(h^t\right) = \frac{a^j\left(h^t\right)}{f\left(z^t\right)}, \quad \hat{\phi}^j\left(h^t\right) = \frac{\phi^j\left(h^t\right)}{f\left(z^t\right)},$$

[2] In this economy, z^t has no effects on aggregate endowments, so it will not affect real allocations - however, agents still trade in securities that are contingent on z^t. If one extends the analysis to arbitrary one-stage Markov processes for aggregate endowment growth. the same type of normalization leads to state-dependent discount rates (unless $\sigma = 1$), but this has little effect on the economic implications of the model.

$$\hat{q}\left(h'|h\right) = q\left(h'|h\right) g\left(z'\right).$$

Then, it is straight-forward to check that $(\hat{C}, \hat{A})$ solve the consumer's problem with the modified allocations and probabilities, if and only if (C, A) solved the original consumer problem. Moreover, these allocations clear the markets if and only if the original allocations do, and given $\hat{Q}$, $\hat{\Phi}$ satisfies (ER) if and only if Φ satisfies (ER) given state-prices Q. The version of our model with i.i.d. shocks to aggregate endowment growth thus maps exactly into the example considered in the paper. The characterization of Proposition 1 (in the paper) then applies to the normalized quantities and prices of the economy with growth.

For the growth version of our model, this implies that $c^j\left(h^t\right) = \overline{c} f\left(z^t\right)$ if $s_t = \mathsf{s}_j$, and $c^j\left(h^t\right) = \underline{c} f\left(z^t\right)$ if $s_t \neq \mathsf{s}_j$, $\phi^j\left(h^t\right) = -\omega f\left(z^t\right)$, and $q\left(h'|h\right) = q_c \hat{\pi}\left(z'\right)/g\left(z'\right)$ if $s' \neq s$, and $q\left(h'|h\right) = q_{nc} \hat{\pi}\left(z'\right)/g\left(z'\right)$ if $s' = s$, where $\overline{c}$, $\underline{c}$, ω, q_c, and q_{nc} are defined as in the paper, for a discount rate $\hat{\beta}$. In particular, this implies that the state-prices divided by endowment growth must add up to 1, or that $\left(\sum_{z'} \hat{\pi}\left(z'\right)/g\left(z'\right)\right)^{-1} = 1$, i.e. that the risk-free real interest rate is given by the harmonic mean of the real growth rate. Thus, in equilibrium (as in the paper), the requirement that debt limits be self-enforcing ties down the risk-free real interest rate at a level that is close to the expected level of the aggregate growth rate.