Decentralized Trading with Private Information*

Michael Golosov
Yale and NES

Guido Lorenzoni
MIT and NBER

Aleh Tsyvinski
Yale and NES

October 2009

Abstract

The paper studies asset pricing in informationally decentralized markets. These markets have two key frictions: trading is decentralized (bilateral), and some agents have private information. We analyze how uninformed agents acquire information over time from their bilateral trades. In particular, we show that uninformed agents can learn all the useful information in the long run and that the long-run allocation is Pareto efficient. We then explore how informed agents can exploit their informational advantage in the short run and provide sufficient conditions for the value of information to be positive. Finally, we provide a numerical analysis of the equilibrium trading dynamics and prices.

*We thank Daron Acemoglu, Fernando Alvarez, Manuel Amador, Abhijit Banerjee, Gadi Barlevy, V.V. Chari, Darrell Duffie, Georgy Egorov, John Geanakoplos, Veronica Guerrieri, Patrick Kehoe, Lasse Pedersen, Dimitri Vayanos, Pierre Olivier Weill, Asher Wolinsky, Pierre Yared and seminar audiences at various universities and conferences for useful comments. We are grateful to our discussants Markus Brunnermeier and Nicolae Garleanu for useful comments and suggestions.

1 Introduction

This paper provides a theory of trading and information diffusion in market environments which are *informationally decentralized*. These markets have two key frictions: trading is decentralized (bilateral) and some agents have private information on the value of the asset traded. Duffie, Garleanu, and Pedersen (2005) started a research agenda of providing a theory of asset pricing in decentralized environments, focusing on the case in which all information is publicly observable.[1] They argue that many important markets such as over-the-counter markets are decentralized. Examples of such markets include markets for mortgage-backed securities, swaps and many other derivatives and real estate markets. In this paper, we study decentralized environments in which some market participants have private information. We analyze how their information gradually spreads in bilateral meetings. In particular, we ask whether all relevant information is revealed over time and whether the informed agents are able to take advantage of their superior information.

Our environment is as follows. Agents start with different endowments of two risky assets, match randomly, and trade in bilateral meetings. Agents are risk averse so there is potential for mutually beneficial trades of the two assets. Before trading begins, a proportion of agents—the informed agents—receive some information about the value of the assets traded. Namely, they observe a signal about the payoffs distribution that determines which one of the two assets is more valuable. In each period, either the game ends with some probability, assets' payoffs are revealed, and the agents consume, or the game continues to the next period. The only information observed by the agents is the history of their matches, but not the portfolios of other agents or their trades. Uninformed agents form beliefs about the value of the two assets based on their own history of trade. This environment is technically and conceptually challenging to analyze because the distribution of asset holdings and the distribution of agents' beliefs about the value of the assets are endogenous and change over time. The evolution of these distributions is crucial in determining the agents' willingness to trade, since it influences their future trading opportunities.

Our first set of results is a theoretical examination of efficiency of equilibria and of the value of information. We first show that the long-run allocations are Pareto efficient. Our argument is as follows. First, we focus on the informed agents and prove that their marginal rates of substitution converge. The intuition for this result is similar to the proof of Pareto efficiency in the decentralized environments with public information. If two informed agents have different marginal rates of substitution, they can always find a trade that improves the utility of both. As their utilities converge to their long run levels, all the potential gains from bilateral trade must be exhausted leading to contradiction. We then show that the marginal rate of substitution for

[1]See also Duffie, Garleanu and Pedersen (2007), Lagos (2007), Lagos and Rochetau (2007), Lagos, Rochetau, and Weill (2007), Vayanos (1998), Vayanos and Weill (2007), and Weill (2007).

the uninformed agents also converge. Our argument is based on finding strategies that allow the uninformed agents to learn the signal received by the informed agents at an arbitrarily small cost. The existence of such strategies implies that either agents eventually learn the signal or this benefit of learning goes to zero. Finally, we show that this implies that equilibrium allocations converge to ex post Pareto efficient allocations in the long run.

We then analyze the value of information in equilibrium. If the initial allocations are not Pareto efficient, i.e., if there are gains from trade,[2] we show that informed agents can receive a higher lifetime utility than uninformed agents. In other words, private information can have a positive value. The intuition is that the uninformed agents will learn the true state of the world only in the long run. Additionally, they have to conduct potentially unprofitable trades in the short run to learn the state of the world. This is why in the short run there can be profitable trading opportunities for the informed agents. We derive sufficient conditions for the value of information to be positive in equilibrium and analyze in detail a simple example, with only one round of trading, where these conditions are satisfied. The example is useful to illustrate the intuition about the strategies and trades of informed and uninformed agents.

The second set of results is based on a numerical analysis of the trading dynamics. We develop an algorithm to compute numerically an equilibrium of the dynamic game and to characterize the evolution of their asset holdings. In particular, we characterize the agents' relative asset positions: the difference between their holdings of the two assets. Since the total supply of the two assets is equal, Pareto efficiency in the long run implies that the relative asset positions of all agents go to zero. That is, all agents converge to a balanced portfolio. Our simulations characterize the short-run behavior of relative asset positions. They show how the trading behavior of informed agents differ depending on their endowment of the valuable asset. The relative asset position of an informed agent who starts with a low endowment of the more valuable asset follows a hump-shaped path. This agent accumulates the valuable asset above its long-run level before the information is revealed. To do so, he mimics the behavior of the uninformed agents and takes advantage of the fact that uninformed agents do not know which asset is more valuable. Once information has diffused in the economy and agents are willing to buy the more valuable asset at a higher price, this agent starts selling it and converges towards a balanced portfolio. The strategy of the informed agent with a large initial endowment of the valuable asset is different. He slowly decumulates his endowment of the valuable asset. His strategy is determined both by signalling considerations and by optimal timing. First, he exchanges small amounts of the valuable asset so as to credibly signal that he attaches a high value to that asset. Second, he makes small trades because he is waiting for information to spread through the economy and agents' to attach a higher value to the asset he holds. Finally, we use our numerical examples to show that it takes longer to converge to efficient allocations

[2] If the initial allocations are already Pareto optimal, a version of Milgrom and Stokey (1982) no trade theorem holds (see Brunnermeier, 2001, for a detailed exposition in various trading environments).

in the environment with private information than in the public information environment.

Our paper is most closely related to Duffie and Manso (2007), Duffie, Giroux, and Manso (2009), Duffie, Malamud, and Manso (2009a, 2009b) and Amador and Weill (2007, 2008). Duffie and Manso (2007), Duffie, Giroux, and Manso (2009), Duffie, Malamud, and Manso (2009a) consider private information trading environments with decentralized markets and focus on information percolation in these environments. They derive important closed form solutions for the dynamics of the trade in an environment where agents observe the information of their match counterparties and derive strong results about the long-run allocations and dynamics. The crucial difference is that they focus on models in which bilateral meetings lead to the immediate transmission of private information between the two participants, while we focus on the problem of strategic information revelation in bilateral trades. In other words, in our environment the rate at which information disseminates through trade is endogenous. Amador and Weill (2007, 2008) and Duffie, Malamud, and Manso (2009b) study dynamics of information dispersion in environments where both private and public information are available. Again, the crucial difference is our focus on strategic issues of information transmission in bilateral trades.

Our work is also closely related to the classic contribution of Wolinsky (1990) on information revelation in pairwise matching environments. The main difference is that in our setting the good is perfectly divisible and agents can trade at endogenously determined prices rather than at fixed terms of trade. These assumptions lead to quite different implications in terms of efficiency. In particular, in our paper, unlike in Wolinsky (1990), information is fully revealed and allocations are ex post efficient in the long run. Intuitively, divisibility allows uninformed agents to strategically experiment by making small, potentially unprofitable trades to learn valuable information. Other related papers in this line of research are Blouin and Serrano (2001), who consider a version of Gale (1987) economy with indivisible good and heterogenous information about its value, and Ostrovsky (2009), who studies information aggregation in dynamic markets with a finite number of partially informed strategic traders. Our analysis of the trades between the informed agents is similar to that of Gale (2000).

The paper is structured as follows. Section 2 describes the environment. Section 3 provides long-run characterization of equilibria. Section 4 contains result on trading and informational rents in the short run. Section 5 contains our numerical analysis. Section 6 concludes. The Appendix contains most of the formal proofs which are sketched in the body of the paper.

2 Setup and trading game

In this section, we introduce the model and define an equilibrium.

2.1 Setup

There are two states of the world $S \in \{S_1, S_2\}$ and two assets $j \in \{1, 2\}$. Asset j is an Arrow security that pays one unit of consumption if and only if state S_j is realized. There is a continuum of agents with von Neumann-Morgenstern expected utility $E[u(c)]$, where E is the expectation operator.

At date 0, each agent is randomly assigned a type i, which determines his initial endowment of the two assets, denoted by the vector $x_{i,0} \equiv (x^1_{i,0}, x^2_{i,0})$. There is a finite set of types N. Each type $i \in N$ is assigned to a fraction f_i of agents. The aggregate endowment of both assets is 1:

$$\sum_{i \in N} f_i x^j_{i,0} = 1 \text{ for } j = 1, 2. \tag{1}$$

We make the following assumptions on preferences and endowments. The first assumption is symmetry. For each agent with a given initial endowment, there is another agent with an endowment that is a mirror image of the first.

Assumption 1. (Symmetry) *For each type $i \in N$ there exists a type $j \in N$ such that the fraction of agents is equal, $f_i = f_j$, and the endowments are such that $(x^1_{i,0}, x^2_{i,0}) = (x^2_{j,0}, x^1_{j,0})$.*

The second assumption imposes usual properties on the utility function. In addition, it requires boundedness from above and a condition ruling out zero consumption in either state.

Assumption 2. *The utility function $u(\cdot)$ is increasing, strictly concave, twice continuously differentiable on R^2_{++}, bounded above, and satisfies $\lim_{c \to 0} u(c) = -\infty$.*

Finally, we assume that the initial endowments are interior.

Assumption 3. *The initial endowment $x_{i,0}$ is in the interior of R^2_+ for all types $i \in N$.*

At date 0, nature draws a binary signal s, which takes the values s_1 and s_2 with equal probabilities. The posterior probability of S_1 conditional on s is denoted by $\phi(s)$. We assume that signal s_1 is favorable to state S_1 and that the signals are symmetric: $\phi(s_1) > 1/2$ and $\phi(s_2) = 1 - \phi(s_1)$. After s is realized, a random fraction α of agents of each type privately observes the realization of s. The agents who observe s are the *informed agents*. Those who do not observe the signal are the *uninformed agents*.

2.2 Trading

After the realization of the signal s, but before the state S is revealed, all agents engage in a trading game set in discrete time. Apart from the presence of asymmetric information, our game is in the tradition of large trading games with bilateral bargaining introduced by Gale (1987).

At the beginning of each period $t \geq 1$, the game continues with probability $\gamma < 1$ and ends with probability $1 - \gamma$. When the game ends, the state S is publicly revealed, and the agents consume the payoffs of their assets.[3]

If the game does not end in a given period, all agents are randomly matched in pairs, and a round of trading takes place. One of the two agents is selected as the *proposer* with probability 1/2. The proposer makes a take it or leave it offer $z = (z^1, z^2) \in R^2$ to the other agent. That is, he offers to deliver z^1 of asset 1 in exchange for $-z^2$ of asset 2. The other agent, the one who receives an offer, is the *responder*. He can accept or reject the offer. Suppose an agent holding the portfolio $x = (x^1, x^2)$ offers a trade z to an agent with the portfolio $\tilde{x} = (\tilde{x}^1, \tilde{x}^2)$. If the offer is accepted, the proposer's portfolio becomes $x - z$, and the responder's portfolio becomes $\tilde{x} + z$. We assume that the proposer can only make feasible offers, $x - z \geq 0$. The responder can only accept an offer if $\tilde{x} + z \geq 0$.[4] If an offer is rejected, both agents keep their portfolios x and $\tilde{x}$. This concludes the trading round.

An agent does not observe the portfolio of his opponent or whether his opponent is informed or not. Moreover, an agent only observes the trading round he is involved in but not the trading rounds of other agents. Therefore, both trading and information revelation take place through decentralized, bilateral meetings.

2.3 Equilibrium definition

We now define a perfect Bayesian equilibrium of the trading game.

We begin by describing individual histories. At date 0, a given agent is assigned the type $i \in N$, which determines his initial endowment. Then he either receives no information or the signal s. That is, he is either an uninformed or an informed agent. The vector $h_0 \in N \times \{U, I_1, I_2\}$ captures the realization of these initial conditions: U stands for uninformed, I_1 and I_2 stand for informed with signal, respectively, s_1 and s_2. Then, in each period $t \geq 1$, the event $h_t = (\iota_t, z_t, r_t)$ is a vector including the following elements: the indicator variable $\iota_t \in \{0, 1\}$, which is equal to 1 if the agent is selected as the proposer and 0 otherwise; the offer made, denoted by $z_t \in R^2$; the indicator $r_t \in \{0, 1\}$, equal to 1 if the offer is accepted and 0 otherwise. The sequence $h^t = \{h_0, h_1, ..., h_t\}$ denotes the history of play up to period t for an individual agent. Let H^t denote the space of all possible histories of length t. Let H^∞ denote the space of all infinite histories, that is, histories along which the game never ends, and let $\Omega = \{s_1, s_2\} \times H^\infty$. A point in Ω describes the whole *potential* history of play for a

[3] Allowing for further rounds of trading after the revelation of S would not change our results, given that at that point only one asset has positive value and no trade will occur. Instead of assuming that the game ends, we can alternatively assume that with probability γ the private signal becomes public information, and interpret utilities of the agents as the equilibrium payoffs from a continuation game with full information. All the results will remain unchanged.

[4] The proposer only observes if the offer is accepted or rejected. In particular, if an offer is rejected the proposer does not know whether it was infeasible for the responder or the responder just chose to reject.

single agent, if the game continues forever. We will use $\left(s, h^t\right)$ to denote the subset of Ω given by all the $\omega = (s, h^\infty) \in \Omega$ such that the truncation of h^∞ at time t is equal to h^t.

We now describe a strategy of a given agent. If an agent is selected as the proposer at time t, his actions are given by the map:

$$\sigma_t^p : H^{t-1} \to \mathcal{P},$$

where $\mathcal{P}$ denotes the space of probability distributions over R^2 with a finite support. That is, we allow for mixed strategies and let the proposer choose the probability distribution $\sigma_t^p\left(\cdot|h^{t-1}\right)$ from which he draws the offer z.[5] If an agent is selected as the responder, his behavior is described by:

$$\sigma_t^r : H^{t-1} \times R^2 \to [0,1],$$

which denotes the probability that the agent accepts the offer $z \in R^2$ for each history h^{t-1}. The strategies are restricted to be feasible for both players. A strategy is fully described by the sequence $\boldsymbol{\sigma} = \{\sigma_t^p, \sigma_t^r\}_{t=1}^\infty$.

We focus on symmetric equilibria where all agents of the same type play the same strategy $\boldsymbol{\sigma}$. Given this strategy, we define a probability measure P on Ω which will be used both to represent ex ante uncertainty from the point of view of a single agent and to capture the evolution of the cross sectional distribution of individual histories in the economy. We say that the probability measure P *is consistent with* $\boldsymbol{\sigma}$, if, for all s and h^{t-1}, $P\left(s, h^{t-1}\right)$ is the probability, conditional on the game ending at time t, that the signal is s and the agent reaches history h^{t-1}, if all agents play $\boldsymbol{\sigma}$. The unconditional probability of the game ending at time t after signal s and history h^{t-1} is then equal to $(1-\gamma)\,\gamma^{t-1} P\left(s, h^{t-1}\right)$. The sequence $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 ... \subset \mathcal{F}$ denotes the filtration generated by the information sets of the agent at the beginning of each period t. The measure P can also be used to characterize the cross sectional distribution of individual histories in a symmetric equilibrium: at the beginning of time t, the mass of agents with history h^{t-1} is equal to $P\left(h^{t-1}|s\right)$.

The probability measure P can be derived recursively in the following manner. The probability of the event $\left(s, h^0\right)$ for each $h^0 \in H^0$ is determined by the exogenous assignment of endowments and information at date 0. For example, $P\left(s_1, h^0\right) = (1/2)\,\alpha f_i$ if $h_0 = (i, I_1)$, given that the probability of s_1 is $1/2$, the agent receives the information I_1 with probability α and the endowment i with probability f_i. Given $P\left(s, h^{t-1}\right)$ for all $h^{t-1} \in H^{t-1}$, the probability $P\left(s, h^{t-1}\right)$ is defined by iterating, as follows. Given that agents are randomly matched, the probability of receiving any offer $z \in R^2$ in period t, for an agent who is not selected as the

[5] We restrict agents to mix over a finite set of offers to simplify the measure-theoretic apparatus. None of the arguments in our proofs require this restriction.

proposer, is equal to[6]

$$\psi_t(z|s) = \int \sigma_t^p\left(z|h^{t-1}\right) dP(\omega|s).$$

Next, we can construct the probability that any offer $z \in R^2$ is accepted in the period t, which is

$$\chi_t(z|s) = \int \sigma_t^r\left(h^{t-1}, z\right) dP(\omega|s).$$

Given $P\left(s, h^{t-1}\right)$, $\psi_t(.|s)$ and $\chi_t(.|s)$, it is then possible to construct $P\left(s, h^t\right)$. Take an agent with history h^{t-1} and suppose that he is selected as the responder, $\iota_t = 0$, receives the offer $z_t = z$, and rejects this offer, $r_t = 0$. His history at the beginning of next period is $h^t = \left(h^{t-1}, (0, z, 0)\right)$ and

$$P\left(s, h^t\right) = \frac{1}{2}\left(1 - \sigma^r\left(h^{t-1}, z\right)\right)\psi_t(z|s) P\left(s, h^{t-1}\right),$$

given that $1/2$ is the probability that the agent is selected as the responder, $\psi_t(z|s)$ is the probability to receive offer z, and $1 - \sigma^r\left(h^{t-1}, z\right)$ is the probability to reject the offer. In a similar way, we have

$$\begin{aligned}
P\left(s, h^t\right) &= \frac{1}{2}\sigma^r\left(h^{t-1}, z\right)\psi_t(z|s) P\left(s, h^{t-1}\right) \text{ if } h_t = (0, z, 1), \\
P\left(s, h^t\right) &= \frac{1}{2}\left(1 - \chi_t(z|s)\right)\sigma^p\left(z|h^{t-1}\right) P\left(s, h^{t-1}\right) \text{ if } h_t = (1, z, 0), \\
P\left(s, h^t\right) &= \frac{1}{2}\chi_t(z|s)\sigma^p\left(z|h^{t-1}\right) P\left(s, h^{t-1}\right) \text{ if } h_t = (1, z, 1).
\end{aligned}$$

Notice that, having restricted agents to randomize over a finite set of offers, $P\left(s, h^t\right)$ will assign positive probability to a finite set of histories h^t.

To assess whether the strategy $\boldsymbol{\sigma}$ is individually optimal, an agent has to form expectations about his opponents' behavior. The agent's beliefs are described by two functions:

$$\begin{aligned}
\delta_t &: H^{t-1} \to [0,1], \\
\delta_t^r &: H^{t-1} \times R^2 \to [0,1],
\end{aligned}$$

which represent, respectively, the probability assigned to signal s_1 after history h^{t-1}, at the beginning of the period, and the probability assigned to signal s_1 after history h^{t-1}, if the agent is the responder and receives offer z. The agent's beliefs are denoted compactly by $\boldsymbol{\delta} = \{\delta_t, \delta_t^r\}_{t=1}^{\infty}$. At each history h^{t-1}, an agent expects that in each period $\tau \geq t$, he will face an opponent with history $\tilde{h}^{\tau-1}$ randomly drawn from the probability distribution $P(\tilde{h}^{\tau-1}|s)$, conditional on s, and he expects his opponent to play the strategy $\boldsymbol{\sigma}$. This completely describes

[6] Given that $\sigma_t^p\left(z|h^{t-1}\right)$ and $\sigma_t^r\left(z|h^{t-1}\right)$ are constant for all $\omega \in \left[s, h^{t-1}\right]$, the integrals in this equation and in the next one can be computed only using our knowledge of $P\left(s, h^{t-1}\right)$.

the agent's expectations about the current and future behavior of other players. For example, the probability distribution of offers expected at time t by an agent at h^{t-1} is given by

$$\psi_t(z|s_1)\,\delta_t\left(h^{t-1}\right) + \psi_t(z|s_2)\left(1 - \delta_t\left(h^{t-1}\right)\right).$$

The beliefs δ_t are required to be consistent with Bayesian updating on the equilibrium path. This implies that

$$\delta_t\left(h^{t-1}\right) = \frac{P(s_1, h^{t-1})}{\sum_s P(s, h^{t-1})},$$

for all histories $h^t \in H^t$ such that $\sum_s P(s, h^{t-1}) > 0$. A similar requirement is imposed on the beliefs δ_t^r.

This representation of the agents' beliefs embeds an important assumption: an agent who observes his opponent play an off-the-equilibrium-path action can change his beliefs about s, but maintains that the behavior of all other agents, conditional on s, is unchanged. That is, he believes that all other agents will continue to play $\boldsymbol{\sigma}$ in the future. This is a reasonable restriction on off-the-equilibrium-path beliefs in a game with atomistic agents and allows us to focus on the agent's beliefs about s, given that s is a sufficient statistic for the future behavior of the agent's opponents.

Notice that informed agents always assign probability 1 to the signal observed at date 0:

$$\begin{aligned}
\delta_t\left(h^t\right) &= \delta_t^r\left(h^t, z\right) = 1 \text{ for all } z \text{ and all } h^t \text{ s.t. } h_0 = (i, I_1),\\
\delta_t\left(h^t\right) &= \delta_t^r\left(h^t, z\right) = 0 \text{ for all } z \text{ and all } h^t \text{ s.t. } h_0 = (i, I_2).
\end{aligned}$$

That is, informed agents do not change their beliefs on the signal s, even after observing off-the-equilibrium-path behavior from their opponents. This fact will play a crucial role in our equilibrium analysis, since it will allow us to derive restrictions on the behavior of informed agents after any possible offer (on and off the equilibrium path).

Given the symmetry of the environment, we will focus on equilibria where strategies and beliefs are *symmetric across states*. To define formally this property, let $\left[h^t\right]^c$ denote the "complement" of history h^t. That is, a history where: (i) the initial endowment is symmetric to the initial endowment in h^t; if the agent is informed I_{-j} replaces I_j; (ii) all the offers received and made at each stage are symmetric to the offers made in h^t, while the responses are the same as in h^t. In particular, if the offer $z = \left(z^1, z^2\right)$ is in h_t, then $z^c = \left(z^2, z^1\right)$ is in $[h_t]^c$. We say that strategy $\boldsymbol{\sigma}$ is symmetric across states if the agent's behavior is identical when we replace asset 1 with asset 2 and h^t with $\left[h^t\right]^c$. That is, $\sigma_t^p(z|h^{t-1}) = \sigma_t^p(z^c|\left[h^{t-1}\right]^c)$ and $\sigma_t^r(h^{t-1}, z) = \sigma_t^r(\left[h^{t-1}\right]^c, z^c)$. For beliefs, we require $\delta\left(h^t\right) = 1 - \delta\left(\left[h^t\right]^c\right)$ and $\delta^r\left(h^t, z\right) = 1 - \delta^r\left(\left[h^t\right]^c, z^c\right)$. This form of symmetry is different from the standard symmetry requirement that all agents with the same characteristics behave in the same manner, which we also assume. The additional restriction is that the agents' behavior does not depend on the labeling of the two states S^1 and S^2.

Throughout the paper, we will use symmetry to mean symmetry across states, whenever there is no confusion.

We now formally define an equilibrium.

Definition 1 *A (perfect Bayesian) symmetric equilibrium is given by a strategy $\boldsymbol{\sigma}$, beliefs $\boldsymbol{\delta}$, and a probability space $(\Omega, \mathcal{F}, P)$, such that:*
(i) $\boldsymbol{\sigma}$ is optimal for an individual agent at each history h^{t-1}, given his beliefs $\boldsymbol{\delta}$ about the signal s and given that he believes that, at each round t, he faces an opponent with history $\tilde{h}^{t-1}$ randomly drawn from $P(\tilde{h}^{t-1}|s)$ who plays the strategy $\boldsymbol{\sigma}$;
(ii) the beliefs $\boldsymbol{\delta}$ are consistent with Bayes' rule whenever possible;
(iii) the probability measure P is consistent with $\boldsymbol{\sigma}$;
(iv) strategies and beliefs are symmetric across states.

Notice that the cross sectional behavior of the economy in equilibrium is purely determined by the signal s. In other words, s is the only relevant aggregate state variable for our trading game, and, for this reason, we will often call it *state* s.

We can now define two stochastic processes, which describe the equilibrium dynamics of individual portfolios and beliefs, conditional on the game not ending. Take the probability space $(\Omega, \mathcal{F}, P)$ and let $x_t(\omega)$ and $\delta_t(\omega)$ denote the portfolio and belief of the agent at the beginning of period t, at ω. Since an agent's current portfolio and belief are, by construction, in his information set at time t, $x_t(\omega)$ and $\delta_t(\omega)$ are $\mathcal{F}_t$-measurable stochastic processes on $(\Omega, \mathcal{F}, P)$.

To establish our results, we will restrict attention to equilibria that satisfy an additional technical property, which we call *uniform market clearing.*

Definition 2 *A symmetric equilibrium satisfies uniform market clearing if for all $\varepsilon > 0$ there is an M such that*

$$\int_{x_t^j(\omega) \leq M} x_t^j(\omega)\, dP(\omega|s) \geq 1 - \varepsilon \text{ for all } t.$$

For a given t, this property is an implication of market clearing and of the dominated convergence theorem. The additional restriction comes from imposing that the property holds uniformly over t. Notice that all equilibria in which the portfolios x_t converge almost surely satisfy uniform market clearing.[7] We will further discuss the role of this assumption when we use it to prove Proposition 2.

2.4 Rational Expectations Equilibrium

Before turning to the analysis of our decentralized trading game, let us briefly discuss a version of our economy with centralized markets, as in, for example, Grossman (1989). The rational

[7] Use Theorem 16.14 in Billingsley (1995).

expectations equilibrium of this economy will provide a useful benchmark for our long run results.

Consider an economy with the same endowments, preferences and information as described above but where, at date 0, agents trade the two assets on a centralized Walrasian market. Let $\delta^I(s)$ denote the belief of an informed agent in state s: $\delta^I(s_1) = 1$ and $\delta^I(s_2) = 0$. Let $\delta^U = 0.5$ denote the belief of an uninformed agent. A *rational expectations equilibrium* consists of prices $q(s) \in R^2_+$ and allocations $\{x_i^*(s,\delta)\}_{i=1}^I$ such that agents optimize:

$$x_i^*(s,\delta) = \arg \max_{q(s)\cdot x \leq q(s)\cdot x_{i,0}} E\{u(x)|q(s),\delta\} \text{ for } s \in \{s_1, s_2\} \text{ and } \delta \in \left\{\delta^U, \delta^I(s)\right\}, \tag{2}$$

and markets clear:

$$\sum_{i\in N} f_i \left[\alpha x_i^{j*}(s,\delta^I(s)) + (1-\alpha)\, x_i^{j*}(s,\delta^U(s))\right] = 1 \text{ for } j = 1,2 \text{ and } s \in \{s_1, s_2\}.$$

In a fully revealing equilibrium $q(s_1) \neq q(s_2)$, so the prices perfectly reveal the signal s to the uninformed agents. This implies that the optimal quantities in (2) are the same for all δ, that is, informed and uninformed with the same endowments get the same equilibrium allocations. To satisfy market clearing it must then be true that the relative price is equal to the ratio of the probabilities:

$$\frac{q^1(s)}{q^2(s)} = \frac{\phi(s)}{1-\phi(s)}.$$

It is easy to verify that a fully revealing equilibrium exists and is ex post Pareto efficient. The second welfare theorem also holds. For any Pareto efficient allocation, there is some initial allocation for which the efficient allocation constitutes a rational expectations equilibrium.

Moreover, it is possible to rule out the existence of non fully revealing equilibria. We now proceed with the proof of this result. The logic of the argument will be useful in the analysis of the economy with decentralized trading (Proposition 2). Suppose there is a non fully revealing equilibrium. Then the relative price $q^1(s)/q^2(s)$ must be the same for both signals s_1 and s_2. The optimality condition for problem (2) gives:

$$\pi(\delta)u'(x_i^{1*}(s,\delta))/(1-\pi(\delta))u'(x_i^{2*}(s,\delta)) = q^1(s)/q^2(s) \tag{3}$$

for $\delta \in \left\{\delta^U, \delta^I(s_1)\right\}$, where $\pi(\delta)$ is the probability that an agent with belief δ assigns to state S_1:

$$\pi(\delta) \equiv \delta\phi(s_1) + (1-\delta)\,\phi(s_2). \tag{4}$$

Now suppose that the relative price satisfies $q^1(s)/q^2(s) \leq 1$. Then condition (3) implies that, conditional on s_1, informed agents will hold more of the first asset than of the second, $x_i^{1*}(s_1,\delta^I(s_1)) > x_i^{2*}(s_1,\delta^I(s_1))$, given that they attach a higher probability to state

S^1, i.e., given that $\pi(\delta^I(s_1))/\left(1-\pi(\delta^I(s_1))\right) > 1$. Similarly, uninformed agents will hold (weakly) more of the first asset than of the second, $x_i^{1*}(s_1, \delta^U(s_1)) \geq x_i^{2*}(s_1, \delta^U(s_1))$, given that

$$\pi(\delta^U(s_1))/\left(1-\pi(\delta^U(s_1))\right) = 1.$$

Therefore, conditional on s_1, total holdings of asset 1 must be greater than total holdings of asset 2. This violates market clearing and gives a contradiction. If the relative price $q^1(s)/q^2(s)$ is greater than 1, proceeding in a symmetric fashion, we can show that total holdings of asset 1 must be smaller than total holdings of asset 2, a contradiction again. Therefore, a non fully revealing equilibrium cannot exist in such centralized environment.

3 Long run characterization

In this section, we provide a characterization of the equilibrium in the long run—i.e., along the path where the game does not end. Our main result is that the equilibrium allocation converges to an ex post Pareto efficient allocation. By ex post Pareto efficient we mean Pareto efficient when the expected utility of each agent is computed conditional on the signal s, that is, after s is publicly revealed but before the state S is revealed.[8] We focus on the long run allocation because at any finite period the allocation will not be Pareto efficient due to the matching friction. Our goal is to understand whether inefficiencies disappear as the game is played for a large number of periods. In the next section, we analyze how trade evolves in the short run, before convergence to efficiency is achieved and how informed agents can obtain informational rents in those periods.

We begin by considering the behavior of informed agents and show that they equalize their marginal rates of substitution in the long run. Then we show that the uninformed agents' marginal rates of substitution also converge to the same value. To prove this result we show that in equilibrium the uninformed agents can always construct small trades that allow them to learn the signal s arbitrarily well. These results are then used to prove that the equilibrium allocation is, in the long run, ex post Pareto efficient (Theorem 1).

3.1 Preliminary considerations

Let us first define and characterize the stochastic process for the agents' expected utility in a symmetric equilibrium. We use the martingale convergence theorem to show that expected utility converges in the long run, conditionally on the game not ending.

[8] This is the standard notion of ex post efficiency, see Holmstrom and Myerson (1983). The same efficiency notion was used above in discussing the Rational Expectations Equilibrium. Notice that after S is revealed all allocations are trivially ex post efficient.

If the game ends, an agent with the portfolio-belief pair (x, δ) receives the expected payoff:

$$U(x, \delta) \equiv \pi(\delta) u(x^1) + (1 - \pi(\delta)) u(x^2),$$

where $\pi(\delta)$ is defined in (4). Using the stochastic processes x_t and δ_t, we obtain a stochastic process u_t for the equilibrium expected utility of an agent if the trading game ends in period t:

$$u_t(\omega) \equiv U(x_t(\omega), \delta_t(\omega)).$$

We can then define the stochastic process v_t as the expected lifetime utility of an agent at the beginning of period t:

$$v_t \equiv (1 - \gamma) E \left[\sum_{s=t}^{\infty} \gamma^{s-t} u_s \mid \mathcal{F}_t \right].$$

In recursive terms, we have

$$v_t = (1 - \gamma) u_t + \gamma E \left[v_{t+1} \mid \mathcal{F}_t \right]. \tag{5}$$

The next lemma establishes that the expected lifetime utility v_t is a bounded martingale and converges in the long run.

Lemma 1 *There exists a random variable $v^\infty(\omega)$ such that*

$$\lim_{t \to \infty} v_t(\omega) = v^\infty(\omega) \;\; a.s.$$

Proof. An agent always has the option to keep his time t portfolio x_t and wait for the end of the game, rejecting all offers and offering zero trades in all $t' \geq t$. His expected lifetime utility under this strategy is equal to u_t. Therefore, optimality implies that

$$u_t \leq E \left[v_{t+1} \mid \mathcal{F}_t \right],$$

which, combined with equation (5), gives

$$v_t \leq E \left[v_{t+1} \mid \mathcal{F}_t \right].$$

This shows that v_t is a submartingale. It is bounded above because the utility function $u(\cdot)$ is bounded above. Therefore, it converges by the martingale convergence theorem. ∎

It is useful to introduce an additional stochastic process, $\hat{v}_t$, which will be used as a reference point to study the behavior of agents who make and receive off-the-equilibrium-path offers. Let $\hat{v}_t$ be the expected lifetime utility of an agent who adopts the following strategy: (i) if selected as the proposer at time t, follow the equilibrium strategy $\boldsymbol{\sigma}$; (ii) if selected as the responder, reject all offers at time t and follow an optimal continuation strategy from $t + 1$ onwards. The expected utility $\hat{v}_t$ is computed at time t immediately after the agent is selected as the proposer or the responder, i.e., it is measurable with respect to $\left(h^{t-1}, \iota_t\right)$, and, by definition, satisfies $\hat{v}_t \leq E\left[v_{t+1} \mid h^{t-1}, \iota_t\right]$.

Recall from the proof of Lemma 1 that u_t is the expected utility from holding the portfolio x_t until the end of the game. The following lemma shows that, in the long run, an agent is almost as well off keeping his time t portfolio as he is under the strategy leading to $\hat{v}_t$.

Lemma 2 *Both u_t and $\hat{v}_t$ converge almost surely to v^∞:*

$$\lim_{t\to\infty} u_t(\omega) = \lim_{t\to\infty} \hat{v}_t(\omega) = v^\infty(\omega) \quad a.s.$$

Proof. As argued in Lemma 1, v_t is a bounded supermartingale and converges almost surely to v^∞. Let $y_t \equiv E[v_{t+1} \mid \mathcal{F}_t]$. Since a bounded martingale is uniformly integrable (see Williams, 1991), we get $y_t - v_t \to 0$ almost surely. Rewrite equation (5) as

$$(1-\gamma)\, u_t = \gamma\left(v_t - E\left[v_{t+1} \mid \mathcal{F}_t\right]\right) + (1-\gamma)\, v_t.$$

This gives

$$u_t - v_t = \frac{\gamma}{1-\gamma}\left(v_t - E\left[v_{t+1} \mid \mathcal{F}_t\right]\right) = \frac{\gamma}{1-\gamma}\left(v_t - y_t\right),$$

which implies $u_t - v_t \to 0$ almost surely. The latter implies $u_t \to v^\infty$ almost surely. Letting $\hat{y}_t \equiv E\left[v_{t+1} \mid h^{t-1}, \iota_t\right]$, notice that $\hat{y}_t \to v^\infty$ almost surely. Since $u_t \leq \hat{v}_t \leq \hat{y}_t$, it follows that $\hat{v}_t \to v^\infty$ almost surely. ■

3.2 Informed agents

We now proceed to characterize the long run properties of the equilibrium. We do not claim that our game has a unique equilibrium. In fact, typically there is a large number of equilibria in this game. Our long run results apply to all the equilibria.

We first focus on informed agents and show that their marginal rates of substitution converge in probability. In particular, we show that, conditional on each signal s, the marginal rates of substitution of all informed agents converge in probability to the same sequence, which we denote $\kappa_t(s)$. In the following, we will refer to $\kappa_t(s)$ as the "long-run marginal rate of substitution" of the informed agents.

The intuition for this result is that if two informed agents have different marginal rates of substitution, they can always find a trade that improves the utility of both. As their utilities converge to their long run levels, all the potential gains from bilateral trade must be exhausted. This implies that marginal rates of substitution must converge. Since this lemma is about the behavior of the informed agents, our argument is similar to the one used to prove Pareto efficiency in decentralized market with full information (Gale, 2000).

Proposition 1 ***(Convergence of MRS for informed agents)*** *There exist two sequences $\kappa_t(s_1)$ and $\kappa_t(s_2)$ such that, conditional on each s, the marginal rates of substitution of informed agents converge in probability to $\kappa_t(s)$:*

$$\lim_{t\to\infty} P\left(\left|\frac{\phi(s)u'(x_t^1)}{(1-\phi(s))\,u'(x_t^2)} - \kappa_t(s)\right| > \varepsilon \mid \delta_t = \delta^I(s), s\right) = 0 \text{ for all } \varepsilon > 0. \tag{6}$$

Proof. We provide a sketch of the proof here and leave the details to the appendix. Without loss of generality, suppose (6) is violated for $s = s_1$. Then, it is always possible to find a period T, arbitrarily large, in which there are two groups, of positive mass, of informed agents with marginal rates of substitution sufficiently different from each other. In particular, we can find a κ^* such that a positive mass of informed agents have marginal rates of substitution below κ^*:

$$\frac{\phi(s_1)\,u'(x_T^1)}{(1-\phi(s_1))u'(x_T^2)} < \kappa^*,$$

and a positive mass of informed agents have marginal rates of substitution above $\kappa^* + \varepsilon$:

$$\frac{\phi(s_1)\,u'(x_T^1)}{(1-\phi(s_1))u'(x_T^2)} > \kappa^* + \varepsilon,$$

for some positive ε. An informed agent in the first group can then offer to sell a small quantity ζ^* of asset 1 at the price $p^* = \kappa^* + \varepsilon/2$, that is, he can offer the trade $z^* = (\zeta^*, -p^*\zeta^*)$. Suppose this offer is accepted and the proposer stops trading afterwards. Then his utility can be approximated as follows:

$$\begin{aligned}
&\phi(s_1)\,u(x_T^1 - \zeta^*) + (1-\phi(s_1))u(x_T^2 + p^*\zeta^*) \\
&\quad\approx u_T + \left[-\phi(s_1)\,u'(x_T^1) + (1-\phi(s_1))p^*u'(x_T^2)\right]\zeta^* \\
&\quad\approx \hat{v}_T + (1-\phi(s_1))u'(x_T^2)\zeta^*\varepsilon/2,
\end{aligned}$$

where we use a Taylor expansion to approximate the utility gain and we use Lemma 2 to show that the continuation utility $\hat{v}_T$ can be approximated by the current utility u_T. By choosing T sufficiently large and the size of the trade ζ^* sufficiently small we can make the approximation errors in the above equation small enough, so that when this trade is accepted it strictly improves the utility of the proposer. All the informed responders with marginal rate of substitution above $\kappa^* + \varepsilon$ are also better off, by a similar argument. Therefore, they will all accept the offer. Since there is a positive mass of them, the strategy described gives strictly higher utility than the equilibrium strategy to the proposer, and we have a contradiction. ■

Two remarks on the argument above: First, there may be uninformed agents who also potentially accept z^*, but this only increases the probability of acceptance, further improving the utility of the proposer. Second, the deviation described (offer z^* at T and stop trading afterwards) is not necessarily the best deviation for the proposer. However, since our argument is by contradiction, it is enough to focus on a simple deviation of this form. We will take a similar approach in many of the following proofs.

3.3 Uninformed agents

We now turn to the characterization of equilibria for uninformed agents. The main difficulty here is that uninformed agents may change their beliefs upon observing their opponent's behavior. Thus an agent who would be willing to accept a given trade ex ante—before updating his beliefs—might reject it ex post. An additional complexity comes from the fact that updated beliefs are not pinned down by Bayes' rule after off-the-equilibrium-path offers, since these offer do not occur in equilibrium. For these reasons, we need a strategy of proof different from the one used for informed agents.

Our argument is based on finding strategies that allow the uninformed agents to learn the signal s at an arbitrarily small cost. This is done in Proposition 3 below. The existence of such strategies implies that either agents eventually learn the signal or the benefit of learning goes to zero. In Theorem 1 we show that this implies that equilibrium allocations converge to ex post Pareto efficient allocations in the long run.

To build our argument, it is first useful to show that in equilibrium the marginal rates of substitution of all agents cannot converge to the same value *independently of the state s*. Since individual marginal rates of substitution determine the prices at which agents are willing to trade, this rules out equilibria in which agents, in the long run, are all willing to trade at the same price, independent of s. The fact that agents are willing to trade at different prices in the two states s_1 and s_2 will be key in constructing the experimentation strategies below. This fact will allow us to construct small trades that are accepted with different probability in the two states. By offering such trades an uninformed agent will be able to extract information on s and thus acquire the information obtained by the informed agents at date 0.

Remember that $\kappa_t(s)$ denotes the long-run marginal rates of substitution of informed agents in state s (Proposition 1). The next proposition shows that in the long run two cases are possible: either the two values $\kappa_t(s_1)$ and $\kappa_t(s_2)$ are sufficiently far from each other, or, in each state s, there must be a sufficient mass of agents with marginal rates of substitution far enough from $\kappa_t(s)$. That is, either the informed agents' marginal rates of substitution converge to different values or there must be enough uninformed agents with marginal rates of substitution different from that of the informed.

Proposition 2 *There exists a period T and a scalar $\bar{\varepsilon} > 0$ such that in all periods $t \geq T$ one of the following must hold: (i) the long-run marginal rates of substitution of the informed* *agents are sufficiently different in the two states:*

$$|\kappa_t(s_1) - \kappa_t(s_2)| \geq \bar{\varepsilon},$$

or (ii) sufficiently many agents have a marginal rate of substitution different from $\kappa_t(s)$:

$$P\left(\left|\frac{\pi(\delta_t)\, u'(x_t^1)}{(1-\pi(\delta_t))\, u'(x_t^2)} - \kappa_t(s)\right| \geq 2\bar{\varepsilon} \mid s\right) \geq \bar{\varepsilon}$$

for $s \in \{s_1, s_2\}$.

The proof of this proposition is in the appendix. The argument is similar in spirit to the one used to rule out non fully revealing equilibria in the Walrasian market of Section 2.4. Recall that in that environment, if prices are independent of s market clearing cannot be satisfied (see p. 11). There is an important difficulty to overcome, when extending this argument to our environment with decentralized trading. In the Walrasian economy, if prices are not revealing, uninformed agents keep their initial beliefs, while in our economy with decentralized trading uninformed agents can always learn something in the early periods of trading. That is, when they reach period t, they have already observed a number of trades, and their beliefs are no longer equal to $\delta^U = 1/2$. In the proof of Proposition 2, we extend the argument by showing that in equilibrium the distribution of beliefs of the uninformed is always biased in the direction of the true signal. That is, when $s = s_1$ there are more uninformed agents with a belief $\delta_t > 1/2$ than uninformed agents with a belief $\delta_t \leq 1/2$. This allows us to show that market clearing is violated if the marginal rates of substitution of all agents are the same and independent of s.

Another, more technical element of the proof of Proposition 2 is the use of the market clearing condition in the long run limit. This is where it is useful to adopt of the uniform market clearing assumption from Definition 2.

3.3.1 Experimentation

Here we show how uninformed agents can experiment and acquire information on the signal s by making small offers. Consider an uninformed agent who assigns probability $\delta \in (0,1)$ to signal s_1 at the beginning of period t. Suppose he makes the offer z and the offer is accepted. Recall that the probability of acceptance of z, conditional on s, is denoted by $\chi_t(z|s)$. Bayes' rule implies that the updated belief δ' of the uninformed agent satisfies

$$\delta' = \frac{\delta \chi_t(z|s_1)}{\delta \chi_t(z|s_1) + (1-\delta)\chi_t(z|s_2)},$$

as long as the offer is accepted with positive probability in some state. If $\chi_t(z|s_1) > \chi_t(z|s_2)$ the acceptance of offer z provides information favorable to s_1 and we have $\delta' > \delta$. Moreover, δ' is larger the larger is the *likelihood ratio* $\chi_t(z|s_1)/\chi_t(z|s_2)$.[9] This ratio captures the informational gain obtained by the uninformed agent. To develop the experimentation strategy, we will construct a sequence of offers such that their likelihood ratio is bounded below by some scalar $\rho > 1$. In this way, if the uninformed agent makes a long enough sequence of offers, and these offers are all accepted, the agent's posterior belief will move sufficiently close to 1.

Sometimes it is not possible to find an offer whose acceptance provides information favorable to s_1. In those cases, however, it is possible to find an offer whose *rejection* is favorable to s_1. That is, sometimes there is no offer such that $\chi_t(z|s_1)/\chi_t(z|s_2) > \rho$ but there is an offer such that $(1 - \chi_t(z|s_1))/(1 - \chi_t(z|s_2)) > \rho$. In general, we will construct a sequence of offers such that, if an uninformed agent makes these offers and receives the appropriate sequence of "yes" and "no", then his belief will get sufficiently close to 1 with sufficiently high probability.

To construct these offers, we will need the following result, which establishes the existence of a trade, of size smaller than θ, with probability of acceptance (or rejection) higher than some positive scalar β, and with likelihood ratio $\chi_t(z|s_1)/\chi_t(z|s_2)$ (or $(1 - \chi_t(z|s_1))/(1 - \chi_t(z|s_2))$) greater than ρ.

Proposition 3 *There are two scalars $\beta > 0$ and $\rho > 1$ with the following property: for all $\theta > 0$ there is a time T such that for all $t \geq T$ there exist a trade z with $\|z\| < \theta$ that satisfies either*

$$\chi_t(z|s_1) > \beta, \qquad \chi_t(z|s_1) > \rho\chi_t(z|s_2), \tag{7}$$

or

$$1 - \chi_t(z|s_1) > \beta, \qquad 1 - \chi_t(z|s_1) > \rho(1 - \chi_t(z|s_2)). \tag{8}$$

Proof. We provide a sketch of the argument here and leave the details to the appendix. We distinguish two cases. By Proposition 2 one of the following must be true in any period t following some period T: (i) either the long-run marginal rates of substitutions of informed agents $\kappa_t(s_1)$ and $\kappa_t(s_2)$ are sufficiently different from each other or (ii) there is a sufficiently large mass of agents with marginal rates of substitution sufficiently different from $\kappa_t(s)$. The proof proceeds differently in the two cases.

Case 1. Suppose that there is a large enough difference between $\kappa_t(s_1)$ and $\kappa_t(s_2)$. Assume without loss of generality that $\kappa_t(s_1) > \kappa_t(s_2)$. Suppose the uninformed agent offers to sell a small quantity ζ of asset 1 at the price $p = (\kappa_t(s_1) + \kappa_t(s_2))/2$, which lies between the two marginal rates of substitutions $\kappa(s_1)$ and $\kappa(s_2)$. That is, he offers the trade $z = (\zeta, -p\zeta)$. We now make two observations on offer z:

Observation 1. In state s_1, there is a positive mass of informed agents willing to accept offer z, provided ζ is small enough and t is sufficiently large. Combining Lemma 2 and Proposition 1, we can show that in state s_1, for t large enough, there is a positive mass of informed agents

[9] In the limit case $\chi_t(z|s_2) = 0$ the likelihood ratio is infinity and δ' is equal to 1.

with marginal rates of substitution sufficiently close to $\kappa_t(s_1)$, who are close enough to their long run utility. These agents are better off accepting z, as they are buying asset 1 at a price smaller than their marginal valuation.

Observation 2. Conditional on signal s_2, the offer z cannot be accepted by any agent, informed or uninformed, except possibly by a vanishing mass of agents. Suppose, to the contrary, that a positive fraction of agents accepted z in state s_2. By an argument symmetric to the one above, informed agents in state s_2 are strictly better off *making* the offer z, if this offer is accepted with positive probability, given that they would be selling asset 1 at a price higher than their marginal valuation (which converges to $\kappa(s_2)$ by Proposition 1). But then an optimal deviation on their part is to make such an offer and strictly increase their expected utility above its equilibrium level, leading to a contradiction.

The first observation can be used to show that the probability of acceptance $\chi_t(z|s_1)$ can be bounded from below by a positive number. The second observation can be used to show that the probability of acceptance $\chi_t(z|s_2)$ can be bounded from above by an arbitrarily small number. These two facts imply that we can make $\chi_t(z|s_1) > \beta$ for some $\beta > 0$ and $\chi_t(z|s_1)/\chi_t(z|s_2) > \rho$ for some $\rho > 1$. So in this case we can always find a trade such that (7) is satisfied, i.e., such that the acceptance of z is good news for s_1. However, when we turn to the next case this will not always be true, and we will need to allow for the alternative condition (8), i.e., rejection of z is good news for s_1.

Case 2. Consider now the second case where the long-run marginal rates of substitution of the informed agents $\kappa_t(s_1)$ and $\kappa_t(s_2)$ are close enough but there is a large enough mass of uninformed agents whose marginal rates of substitution is far from $\kappa_t(s_1)$, conditional on s_1.

This means that we can find a price p such that the marginal rates of substitution of a group of uninformed agents are on one side of p and the long-run marginal rates of substitution of informed agents $\kappa_t(s_1)$ and $\kappa_t(s_2)$ are on the other side. Consider the case where the MRS of a group of uninformed agents is greater than p, and $\kappa_t(s_1)$ and $\kappa_t(s_2)$ are smaller than p (the other case is symmetric). Then the uninformed agents in this group can make a small offer to buy asset 1 at a price p and the informed will accept this offer conditional on both signals s_1 and s_2. If the probabilities of acceptance conditional on s_1 and s_2 were sufficiently close to each other, this would be a profitable deviation for the uninformed, since then their ex post beliefs would be close to their ex ante beliefs. In other words, in contrast to the previous case, the uninformed would have a utility gain but would not learn from the trade. It follows that the probabilities of acceptance of this trade must be sufficiently different in the two states s_1 and s_2. This leads to either (7) or (8), completing the proof. ■

3.3.2 Convergence of marginal rates of substitution

We now characterize the properties of the long-run marginal rates of substitution of uninformed agents. The next proposition shows that the convergence result established for informed agents

(Proposition 1) extends to uninformed agents.

In what follows, instead of looking at the ex ante marginal rate of substitution, given by $\pi(\delta_t)u'(x_t^1)/\left(1-\pi(\delta_t)\right)u'(x_t^2)$, we establish convergence for the ex post marginal rate of substitution $\phi(s)u'(x_t^1)/\left(1-\phi(s)\right)u'(x_t^2)$. This is the marginal rate of substitution at which an agent would be willing to trade asset 2 for asset 1 *if he could observe the signal* s. As we will see, this is the appropriate convergence result given our objective, which is to establish the ex post efficiency of the equilibrium allocation.

Proposition 4 ***(Convergence of MRS for uninformed agents)*** *Conditional on each* s*, the marginal rate of substitution of any agent, evaluated at the full information probabilities* $\phi(s)$ *and* $1-\phi(s)$*, converges in probability to* $\kappa_t(s)$*:*

$$\lim_{t\to\infty} P\left(\left|\frac{\phi(s)u'(x_t^1)}{\left(1-\phi(s)\right)u'(x_t^2)}-\kappa_t(s)\right|>\varepsilon \mid s\right)=0 \text{ for all } \varepsilon>0. \tag{9}$$

Note that in equation (9) we use the full information probabilities $\phi(s)$ rather than the belief $\pi(\delta_t)$ that an agent with allocation x_t might have.

Proof. We provide a sketch of the proof, leaving the details to the appendix. Suppose condition (9) fails to hold. Without loss of generality, we focus on the case where (9) fails for $s=s_1$. This means that there is a period T in which with a positive probability an uninformed agent has ex post marginal rate of substitution sufficiently far from $\kappa_T(s_1)$ and is sufficiently close to his long run utility. Without loss of generality, suppose his marginal rate of substitution is larger than $\kappa_T(s_1)$. To reach a contradiction, we construct a profitable deviation for this agent.

Before discussing the deviation, it is useful to clarify that, at time T, the uninformed agent has all the necessary information to check whether he should deviate or not. He can observe his own allocation x_T, compute $\phi(s_1)u'(x_T^1)/\left(1-\phi(s_1)\right)u'(x_T^2)$, and verify whether this quantity is sufficiently larger than $\kappa_T(s_1)$ (which is known, since it is an equilibrium object).

The deviation then consists of two stages:

Stage 1. This is the experimentation stage, which lasts from period T to period $T+J-1$. As stated in Proposition 3, the agent can construct a sequence of small offers $\{\hat{z}_j\}_{j=0}^{J-1}$ such that, if these offers are followed by the appropriate responses, the agent's ex post belief on signal s_1 will converge to 1. To be precise, for this to be true it must be the case that the agent does not start his deviation with a belief δ_T too close to 0. Otherwise, a sequence of J signals favorable to s_1 is not enough to bring δ_{T+J} sufficiently close to 1. Therefore, when an agent starts deviating we also require δ_T to be larger than some positive lower bound $\underline{\delta}$, appropriately defined.

Stage 2. At date $T+J$, if the agent has been able to make the whole sequence of offers $\{\hat{z}_j\}_{j=0}^{J-1}$ and has received the appropriate responses (that is, the responses which bring the probability of s_1 close to 1), he then makes one final offer z^*. This is an offer to buy a small quantity ζ^* of asset 1 at a price p^*, which is in between the agent's own marginal rate of substitution and $\kappa_T(s_1)$. By choosing T large enough, we can ensure that there is a positive mass of informed agents close enough to their long-run marginal rate of substitution, who are willing to sell asset 1 at that price.[10] Therefore, the offer is accepted with a positive probability. The utility gain for the uninformed agent, conditional on reaching Stage 2 and conditional on z^* being accepted, can be approximated by

$$U(x_T + \sum_{j=0}^{J-1} \hat{z}_j + z^*, 1) - U(x_T, 1),$$

given that, after the experimentation stage the agent's ex post belief approaches 1. Moreover, by making the final offer z^* and the experimenting offers $\hat{z}_j$ sufficiently small, this utility gain can be approximated by

$$U(x_T + z^*, 1) - U(x_T, 1) \approx \phi(s) u'(x_T^1)\zeta^* - (1 - \phi(s))\, u'(x_T^2) p^* \zeta^* > 0.$$

The last expression is positive because p^* was chosen smaller than the marginal rate of substitution $\phi(s)u'(x_T^1)/\,(1 - \phi(s))\, u'(x_T^2)$. In the appendix we show that this utility gain is large enough that the deviation described is ex ante profitable, i.e., it is profitable from the point of view of period T. To do so, we must ensure that the utility losses that may happen along the deviating path (e.g., when some of the experimenting offers do not generate a response favorable to s_1 or when the agent is not selected as the proposer) are small enough. To establish this, we use again the fact that the experimenting offers are small. As usual, the argument in the appendix makes use of the convergence of utility levels in Lemma 2, to show that a utility gain relative to the current utility u_t, leads to a profitable deviation relative to the the expected utility $\hat{v}_t$. Since we found a profitable deviation for the uninformed agents, a contradiction is obtained which completes the argument. ■

3.4 Efficiency

Having characterized the portfolios of informed and uninformed agents in the long run, we can finally derive our key efficiency result.

Theorem 1 *All symmetric equilibrium allocations which satisfy uniform market clearing con-*

[10] Notice that the uninformed agent is using $\kappa_T(s_1)$ as a reference point for the informed agents' marginal rate of substitution, while offer z^* is made in period $T + J$. Lemma 8 in the Appendix ensures that $\kappa_T(s_1)$ and $\kappa_{T+J}(s_1)$ are sufficiently close, so that at $T + J$ enough informed agents have marginal rate of substitution near $\kappa_T(s_1)$.

verge to ex post efficient allocations in the long run, i.e.,

$$\lim_{t\to\infty} P\left(\left|x_t^1 - x_t^2\right| > \varepsilon\right) = 0 \text{ for all } \varepsilon > 0. \tag{10}$$

The long-run marginal rates of substitution $\kappa_t(s)$ converge to the ratios of the conditional probabilities of states S_1 and S_2:

$$\lim_{t\to\infty} \kappa_t(s) = \phi(s)/(1-\phi(s)) \text{ for all } s \in \{s_1, s_2\}. \tag{11}$$

Proof. We provide a sketch of the proof and leave the formal details to the appendix. First, suppose that $\kappa_t(s) > (1+\varepsilon)\,\phi(s)/(1-\phi(s)$ for some $\varepsilon > 0$, for infinitely many periods. Then Proposition 4 can be used to show that the agents' holdings of asset 1 will be larger than their holdings of asset 2. This, however, violates market clearing. In a similar way, we rule out the case in which $\kappa_t(s) < (1-\varepsilon)\,\phi(s)/(1-\phi(s))$ for some $\varepsilon > 0$, for infinitely many periods. This proves (11). Then, using this result and Proposition 4, we can show that $u'(x_t^1)/u'(x_t^2)$ converges in probability to 1, which implies (10). ■

This theorem establishes that, equilibrium allocations converge to ex post Pareto efficient allocations. This implies that the long run equilibrium allocation coincides with a rational expectation equilibrium defined in Section 2.4 for *some* initial allocations. It does not show that starting with the *same* initial allocations our decentralized trading game converges to the *same* long run outcomes as the rational expectation equilibrium of the centralized Walrasian market. In fact, we will see that this is typically not the case. Remember that in the centralized environment informed and uninformed agents with the same initial endowment reach the same equilibrium allocation. This will not be the case in the decentralized environment, where informed agents can reach, on average, a higher expected utility, as we show in the next section.

We conclude this section with a brief comparison of our analysis with Wolinsky (1990). In our model, in the long run all agents are only willing to trade at a single price (conditional on the state s) which is the same as the Walrasian REE price $\phi(s) / (1-\phi(s))$. Wolinsky (1990) also analyzes a dynamic trading game with asymmetric information and shows that in steady state different trades can occur at different prices, so a fraction of trades can occur at a price different from the Walrasian REE price. The approaches in the two papers are quite different as Wolinsky (1990) considers a game where a fraction of traders enter and exit the game at each point time, focuses on steady-state equilibria, and takes limits as discounting goes to zero, while we look at a game with a fixed set of participants and a fixed "discount rate" (our probability γ) and look at long-run outcomes (in this section). However, the crucial difference is that Wolinsky (1990) features an indivisible good which can only be traded once, while we have perfectly divisible goods (assets) which are traded repeatedly. This makes the process of experimentation by market participants very different in the two environments. In Wolinsky

(1990) agents only learn if their offers are rejected, once their offer is accepted they trade and exit the market. In our environment, agents keep learning and trading along the equilibrium play, and, in particular, can learn by making small trades (as shown in Proposition 3) and then use the information acquired to make Pareto improving trades with informed agents (as shown in Proposition 4).

4 Trading and informational rents in the short run

We now turn to analyze the short run properties of the equilibrium focusing on two related issues: (i) how informed agents obtain informational rents, and (ii) how asymmetric information distorts equilibrium trading in the short run. First, we define formally the value of information and provide sufficient conditions for the value of information to be positive in equilibrium. Second, we turn to a simple static example that illustrates this result and shows how the presence of informed traders distorts equilibrium trading. The static example sets the stage and provides useful intuition for the numerical analysis in the next section.

4.1 Value of information

To define the value of information, consider an agent who reaches period t with the portfolio $x \in R_+^2$ and belief $\delta \in [0,1]$. From the point of view of individual optimization these are the only relevant state variables, so we can use the value function $V_t(x,\delta)$ to denote his expected payoff. Notice that the value function is defined for all possible pairs (x,δ), not only for those that arise on the equilibrium path with positive probability. Suppose the agent is allowed to observe the signal s at time t. Then his expected utility would increase by

$$\mathcal{I}_t(x,\delta) \equiv \delta V_t(x,1) + (1-\delta)V_t(x,0) - V_t(x,\delta),$$

given that the agent can re-optimize after observing s and updating his belief to either 1 or 0. We call $\mathcal{I}_t(x,\delta)$ the *value of information.* Since after observing s an agent always has the option to follow his original strategy, the value of information is always nonnegative, $\mathcal{I}_t(x,\delta) \geq 0$. In the Rational Expectations Equilibrium of Section 2.4 the value of information is zero. There uninformed agents learn instantaneously the value of s from their observation of equilibrium prices and informed agents have no informational advantage when trading. We want to investigate whether the value of information is positive in our decentralized trading environment, where uninformed agents learn gradually the value of s from the outcome of their bilateral meetings.

First, consider the case in which the economy starts at an efficient allocation, that is, all agents begin with equal endowments of the two assets, $x_{0,i}^1 = x_{0,i}^2$ for all i. In this case, the no trade theorem of Milgrom and Stokey (1982) applies and no trade is the only equilibrium.

The value of information is then zero, as agents cannot gain from changing their equilibrium trading strategy after learning s.

Theorem 2 *(Milgrom and Stokey) Suppose $x^1_{0,i} = x^2_{0,i}$ for all i. Then there is no trade in equilibrium and the equilibrium value of information is always zero, i.e., $\mathcal{I}_t(x_t, \delta_t) = 0$, almost surely, for all t.*

Proof. The proof is a direct adaptation of Theorem 1 in Milgrom and Stokey (1982) and is omitted. ■

Next, consider the case in which the initial allocation is not Pareto efficient, that is, when agents begins with different endowments of the two assets ($x^1_{0,i} \neq x^2_{0,i}$ for some i). First, notice that in the long run we do not expect the value of information to be positive. The characterization in Theorem 1 shows that in the long run an uninformed agent will exhaust all the utility gains that he can obtain by observing the signal s. In particular, after observing s, he cannot gain from trading with informed agents, as they are willing to trade at the price $\phi(s) / (1 - \phi(s))$ which is equal to his own ex post marginal rate of substitution. Moreover, the logic of the no trade theorem suggests that he won't be able to gain from trading with uninformed agents, given that everybody knows that there are no mutual gains from trade to be obtained. Therefore, we focus on finding positive informational rents in the short run.

When agents begin with different endowments, some trade needs to occur in the short run as agents converge towards their long run allocations. As trade takes place, informed agents may be able to exploit their superior knowledge about the willingness to trade of their informed trading partners in states s_1 and s_2 to obtain higher expected utility. The following result provides sufficient conditions for the value of information to be positive in equilibrium. These conditions require: (a) the differentiability of the value functions for informed agents; (b) that the marginal value of asset 1 be higher for an informed agent after observing signal s_1 than after observing signal s_2, and the reverse for asset 2. We do not have general conditions on primitives to ensure that (a) and (b) are satisfied in equilibrium. In the example below, we show that they are satisfied and discuss their role.[11]

Theorem 3 *Suppose there are two (symmetric) types with $x^1_{0,1} \neq x^2_{0,1}$ and suppose that in equilibrium the value functions $V_t(x, 0)$ and $V_t(x, 1)$ are differentiable in x and satisfy*

$$\frac{\partial V_t(x,1)}{\partial x^1} > \frac{\partial V_t(x,0)}{\partial x^1}, \qquad \frac{\partial V_t(x,1)}{\partial x^2} < \frac{\partial V_t(x,0)}{\partial x^2}, \tag{12}$$

for all $x \in R^2_{++}$. Then the value of information is positive in period 0:

$$\mathcal{I}_0(x_{0,1}, 1/2) > 0.$$

[11]Notice that the differentiability of the value function is not easy to establish in general, because the feasible set in each round of trading is not in general convex, so the value function is not in general concave.

The proof of the theorem is in the appendix. Before discussing the intuition behind this result, it is useful to introduce a simple static example. We will use the example to construct explicitly an equilibrium where information has positive value. Then we will see that the conditions of Theorem 3 apply, so the value of information must be positive in all equilibria.

In Theorem 3 we restrict attention to the case of two endowment-types. This assumption helps in the proof because it allows us to characterize the set of accepted offers when agents are at their initial endowments. In particular, with only two (symmetric) endowment-types, one type will only be concerned with attracting the other type. This allows us to show that the set of accepted offers has a differentiable frontier, which is an important step in the proof of the theorem. We will return to this point after discussing the example.

4.2 A static example

Consider a special case of our trading game.[12] Set γ equal to zero, so agents engage only in one round of trading after which the game ends. Moreover, assume there are two types of agents, $N = 2$, with endowments $x_{0,1} = (e_h, e_l)$ and $x_{0,2} = (e_l, e_h)$, with $e_h > e_l > 0$ and $e_l + e_h = 1$. Finally, assume that the fraction α of informed agents is negligible, so their presence does not affect the strategies of uninformed agents.

This game, as the more general setup above, has many equilibria, supported by different out-of-equilibrium beliefs. While the theoretical results above apply to all equilibria, here, we focus on a specific equilibrium, choosing one which satisfies the intuitive criterion of Cho and Kreps (1988). Figures 1 illustrates the equilibrium construction using an Edgeworth box.

We first construct the strategies and beliefs of the agents and then verify that they constitute an equilibrium. We begin by setting up two optimization problems which are used to construct our candidate strategies for informed and uninformed proposers. By symmetry, we can limit our attention to the strategies of proposers with endowment (e_h, e_l). An uninformed proposer with this endowment makes the offer z^*. Since there is a negligible mass of informed agents, the offer z^* is received with equal probability (one) in the two states of the world s_1 and s_2. The belief of an uninformed responder who receives offer z^* is then equal to $\delta = 1/2$. The proposer only cares about his offer being accepted by uninformed responders with symmetric endowment (e_l, e_h) and belief $\delta = 1/2$ because: (i) the probability of meeting an informed agent is negligible, and (ii) if the proposer is matched with an uninformed agent with his same endowment (e_h, e_l) and belief $\delta = 1/2$ there is no mutually beneficial trade. Therefore, trade z^* is chosen to maximize the after-trade utility of the uninformed proposer:

$$\max_{z^1,\, z^2} \frac{1}{2} u(e_h - z^1) + \frac{1}{2} u(e_l - z^2),$$

[12] The setup of Rocheteau (2008) shares some similarities with this static example in a model with money and other assets.

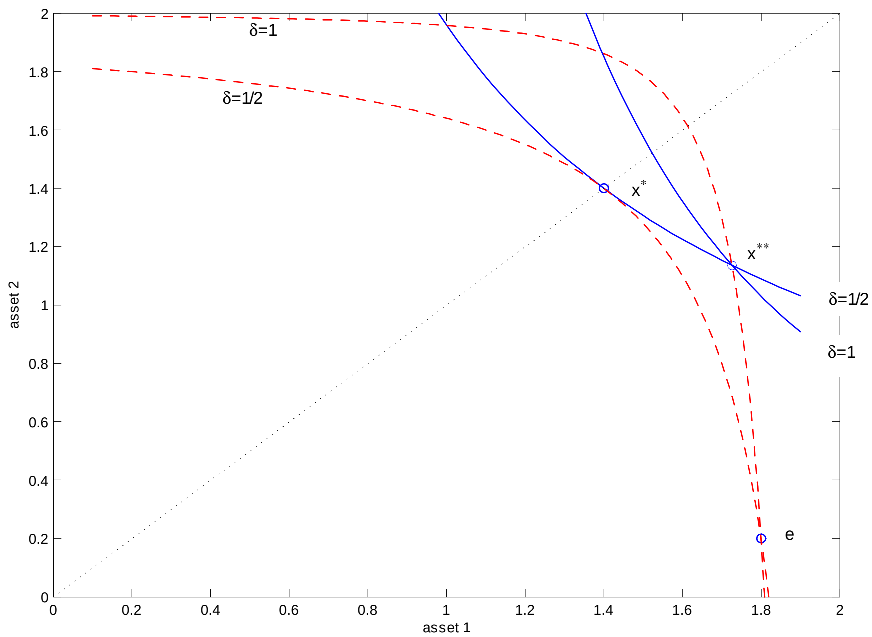

Figure 1: Static example: equilibrium allocations

subject to the constraint

$$\frac{1}{2}u(e_l + z^1) + \frac{1}{2}u(e_h + z^2) \geq \frac{1}{2}u(e_l) + \frac{1}{2}u(e_h), \tag{13}$$

which ensures that the offer is accepted by an uninformed responder with endowment (e_l, e_h) and belief $\delta = 1/2$.

The solution to the maximization problem above is illustrated graphically in the Edgeworth box of Figure 1. The two assets are measured on the two axes, with the proposer's origin corresponding to the lower left corner, and the responder's origin corresponding with the upper right corner. Point e correspond to the initial endowment for two symmetric agents with (e_h, e_l) and (e_l, e_h). The indifference curve of the uninformed responder is the dashed red line labelled $\delta = 1/2$ that goes through the endowment point. Point $x^* = (e_h, e_l) + z^*$ is where this curve is tangent to the indifference curve of the uninformed proposer, the blue line labelled $\delta = 1/2$. At the optimum, both agents equalize their holdings of the two assets.

Turning to the informed proposers, two cases are possible. The proposer may be a *rich informed agent*, who holds a larger endowment of the asset that is more likely to pay off. This is the case of an informed proposer with endowment (e_h, e_l) in state s_1. Or he may be a *poor informed agent*, who holds a smaller endowment of the asset which is more likely to pay off.

This is the case of an informed proposer with endowment (e_h, e_l) in state s_2.

In our equilibrium, a rich informed proposer with endowment (e_h, e_l) makes an offer $z^{**} \neq z^*$ that fully reveals his information. We will see that this is optimal for him, since it allows him to sell his asset at a higher price. The belief of an uninformed responder after observing z^{**} is $\delta = 1$. We then choose z^{**} to be the offer that maximizes the proposer expected utility:

$$\max_{z^1,\, z^2} \phi(s_1)\, u(e_h - z^1) + (1 - \phi(s_1))u(e_l - z^2), \tag{14}$$

subject to

$$\phi(s_1)\, u(e_l + z^1) + (1 - \phi(s_1))u(e_h + z^2) \geq \phi(s_1)\, u(e_l) + (1 - \phi(s_1))u(e_h), \tag{15}$$

and

$$\frac{1}{2}u(e_h - z^1) + \frac{1}{2}u(e_l - z^2) \leq \frac{1}{2}u(e_h - z^{1*}) + \frac{1}{2}u(e_l - z^{2*}). \tag{16}$$

Constraint (15) ensures that an uninformed responder with endowment (e_l, e_h) is willing to accept the trade after updating his belief to $\delta = 1$. Constraint (16) ensures that an uninformed proposer does not prefer offering z^{**} to offering z^*. That is, it ensures that uninformed proposers do not try to mimic the rich informed agents to sell their asset at a high price. The solution to this problem is also illustrated in Figure 1. The dashed red line labelled $\delta = 1$ corresponds to constraint (15) and the solid blue line labelled $\delta = 1/2$ corresponds to constraint (16). The optimum corresponds to point x^{**}.

Finally, a poor informed proposer with endowment (e_h, e_l) mimics the behavior of the uninformed proposers with the same endowment and makes offer z^*. Since this agent is selling the less valuable asset, he is better off hiding his information.

We now proceed to describe beliefs and strategies of uninformed responders. The strategies of informed responders can be easily derived, but, given that there is a negligible measure of them, we can ignore them as they do not affect the strategies of the other agents.

We begin with the beliefs. We have already shown that beliefs on the equilibrium path must be equal to $\delta = 1/2$ after receiving offer z^*, which reveals no information on the state s, and $\delta = 1$ after receiving z^{**}, which is perfectly revealing. To complete the description of beliefs, both on and off the equilibrium path, we assume that uninformed responders keep their beliefs unchanged at $\delta = 1/2$ except in the following two cases. First, if they receive an offer z that satisfies

$$\phi(s_1)\, u(e_h - z^1) + (1 - \phi(s_1))\, u(e_l - z^2) > \phi(s_1)\, u(e_h - z^{1*}) + (1 - \phi(s_1))\, u(e_l - z^{2*})$$

and (16) then they assign probability one to s_1. Second, if they receive an offer that satisfies

$$\phi(s_2)\, u(e_h - z^1) + (1 - \phi(s_2))\, u(e_l - z^2) > \phi(s_2)\, u(e_h - z^{1*}) + (1 - \phi(s_2))\, u(e_l - z^{2*})$$

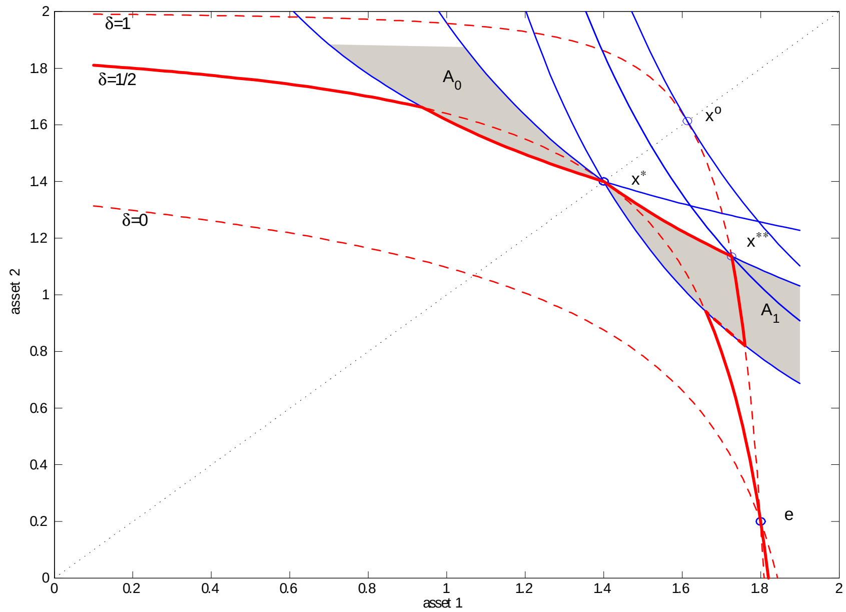

Figure 2: Static example: checking optimality

and (16) then they assign probability zero to s_1.

Given the beliefs just described, the strategy of uninformed responders is simply to accept offer z if and only if it satisfies

$$\pi(\delta)\, u(e_l + z^1) + (1 - \pi(\delta))\, u(e_h + z^2) \geq \pi(\delta)\, u(e_l) + (1 - \pi(\delta))\, u(e_h).$$

We now verify that the strategies and beliefs just defined constitute an equilibrium. By construction, the beliefs of the responders are consistent with Bayes' rule and their strategy is optimal. So we only need to check that the proposers' strategies are optimal.

Since a proposer faces a negligible fraction of informed responders, his behavior is affected only by the uninformed responders. Figure 2 is an Edgeworth box like Figure 1. Given the beliefs described above, we identify three sets of offers: those corresponding to region A_1 lead to $\delta = 1$; those corresponding to region A_0 lead to $\delta = 0$; all the others lead to $\delta = 1/2$. The three dotted red curves correspond to the uninformed responder's indifference curves going through the endowment point, for beliefs $\delta = 0$, $\delta = 1/2$, and $\delta = 1$. This implies that the feasible set for the proposer is the region delimited by the thick red line (below and to the left of that line).

We can now check optimality. There are three blue lines going through point x^*, representing the indifference curves of the proposer for beliefs $\delta = 0$, $\delta = 1/2$, and $\delta = 1$. The indifference curve with an intermediate slope is the one corresponding to an uninformed proposer with $\delta = 1/2$. Offering z^* and reaching x^* is optimal for this agent. Next, the steepest indifference curve corresponds to a rich informed proposer, with $\delta = 1$. This agent prefers any offer in the set A_1 to offer z^*, since he gets a higher price for the asset he is selling. So offering z^{**} and reaching x^{**} is optimal for him. Finally, the flattest indifference curve corresponds to a poor informed proposer, with $\delta = 0$. This agent would get a higher utility than he gets at x^* if he could convince the responder to accept any offer in A_0. However, there are no offers that lie in A_0 *and* are to the left of the responder's indifference curve labelled $\delta = 0$. Thus, it is optimal for the poor informed proposer to mimic the uninformed and offer z^*. It can be checked that this equilibrium satisfies Cho and Kreps (1987) intuitive criterion.

It is useful to compare this equilibrium with the one arising in the same game with complete information, where the signal s is observed by all agents. In this case, in state s_1 the indifference curves of both proposer and responder are steeper and the unique subgame perfect equilibrium corresponds to the trade which maximizes the proposer's expected utility, while leaving the responder indifferent between accepting and rejecting, leading to allocation x^o. This is an ex post Pareto efficient allocation which lies on the 45 degree line, where agents equalize their holdings of the two assets. In contrast, in the game with asymmetric information, when a rich informed agent meets an uninformed responder, they reach x^{**} which is not an ex post Pareto efficient allocation. Why is the allocation x^o not an equilibrium? The reason is that if it was, both informed and uninformed agents would prefer to make the offer leading to x^o. Hence, the agent receiving such offer would not change his beliefs, thus remaining on the indifference curve with $\delta = 1/2$ and rejecting the offer.

The trading prices can be evaluated graphically by taking the vectors connecting the endowment point to the final allocations x^* and x^{**} and looking at their slopes.[13] Notice that the price of asset 1 offered by a rich informed agent is higher than: (a) the price offered by uninformed agents, and (b) the price that informed agents would offer in the game with complete information. This example shows how informed agents can receive a higher payoff than uninformed agents. By offering z^{**} instead of z^*, the rich informed agent can credibly reveal his information and sell a smaller amount of asset 1 at a higher price. The reason is that only an agent who observes signal s_1 is willing to retain a portfolio so unbalanced in favor of asset 1. On the other hand, as discussed above, this signalling leads to ex post inefficiency.

This example illustrates several important features of equilibrium trading, which, as we will see, remain true in the computed equilibrium of the dynamic game. First, since rich informed agents need to signal their information, it takes them longer to reach efficient outcomes than with full information. Second, rich informed agents generally prefer to sell little of their endowment early on in the game and rebalance their portfolio slowly. Finally, poor informed agents sell their holdings of the less valuable asset, taking advantage of the fact that uninformed

[13] To make the figure more readable, the trade vectors are not drawn.

agents do not know which asset is more valuable.

So far we have concentrated on a specific equilibrium of our example. However, the mechanism generating informational rents is present in *any* equilibrium. To see this, let us go back to Theorem 3 and discuss its application to our example. In the static economy of the example, the value function $V_1(x, \delta)$ is equal to the expected utility $U(x, \delta)$. Therefore, the differentiability requirements for Theorem 3 are immediately satisfied and so are conditions (12) because

$$\frac{\partial U(x,1)}{\partial x^1} = \phi(s_1) u'\left(x^1\right) > \phi(s_2) u'\left(x^1\right) = \frac{\partial U(x,0)}{\partial x^1} \tag{17}$$

and

$$\frac{\partial U(x,1)}{\partial x^2} = (1 - \phi(s_1)) u'\left(x^2\right) > (1 - \phi(s_2)) u'\left(x^2\right) = \frac{\partial U(x,0)}{\partial x^2} \tag{18}$$

follow from $\phi(s_1) > \phi(s_2)$. Therefore, Theorem 3 applies and the value of information is always positive.

To complete this section, we provide some intuition for Theorem 3, sketching the argument of the proof in the context of our example. Suppose, by contradiction that the value of information is zero. Consider an offer z made by an uninformed proposer and accepted with positive probability by some agent (in the Appendix we show that such an offer must exist). Now if the proposer has endowment (e_h, e_l) it can be argued that offer z will be accepted only by agents with symmetric endowment (e_l, e_h) (the formal details are in the Appendix). This means that the set of agents who accept z is characterized as the set of agents for which

$$U((e_l, e_h) + z, \delta) \geq U((e_l, e_h), \delta) \tag{19}$$

for all beliefs δ in some interval $\left[\delta', \delta''\right]$. Suppose, without loss of generality, that $z^1 > 0$ and $z^2 < 0$, i.e., the proposer is offering to sell asset 1. Conditions (17) and (18) can then be used to show that the acceptance condition (19) will hold as a strict inequality except for the most optimistic agent with belief δ''. In other words, if we perturb the offer z, taking an offer $\tilde{z}$ near z such that the offer is accepted by an agent with belief δ'', i.e., such that

$$U((e_l, e_h) + \tilde{z}, \delta'') \geq U((e_l, e_h), \delta''), \tag{20}$$

then offer $\tilde{z}$ will be accepted at least by all agents who accept z. In turn, this implies that local perturbations of z that satisfy (20) cannot dominate z for the proposer. This means that the marginal rate of substitution of the proposer (after z is accepted) must be equal to

$$\frac{\partial U((e_l, e_h) + z, \delta'')/\partial x^1}{\partial U((e_l, e_h) + z, \delta'')/\partial x^2}. \tag{21}$$

Now we can use our assumption of zero value of information to reach a contradiction. If making offer z is optimal for an uninformed proposer and the value of information is zero, it must also be optimal to offer z for an informed proposer with the same endowment and belief $\delta = 1$. But the marginal rate of substitution of these two proposers cannot be both equal to the same value (21), given that they have different beliefs. This follows from conditions (17) and (18), which imply that the two marginal rates of substitution satisfy

$$\begin{aligned} \frac{\partial U((e_h, e_l) - z, 1/2)/\partial x^1}{\partial U((e_h, e_l) - z, 1/2)/\partial x^2} &= \frac{u'\left(e_h - z^1\right)}{u'\left(e_l - z^2\right)} \neq \\ \frac{\phi\left(s_1\right) u'\left(e_h - z^1\right)}{\left(1 - \phi\left(s_1\right)\right) u'\left(e_l - z^2\right)} &= \frac{\partial U((e_h, e_l) - z, 1)/\partial x^1}{\partial U((e_h, e_l) - z, 1/2)/\partial x^2}. \end{aligned}$$

Notice that the assumption of two endowment-types is useful in the argument above, when we construct the set of local perturbations $\tilde{z}$ satisfying (20). In particular, with two types we immediately see that this set has a differentiable frontier, which allows us to derive our contradiction. In fact, with more than two types, it is possible to construct counterexamples where the equilibrium offers are at a kink of the frontier of accepted offers. In those examples, the value of information is zero when the signal s is not informative enough (i.e., when $\phi\left(s_1\right)$ is near 1/2). We believe that in the fully dynamic version of our model (i.e., with $\gamma > 0$) it might be possible to develop alternative arguments to prove that the value of information is always positive, with any number of types, exploiting the convergence properties of marginal rates of substitution. The main difficulty with this type of arguments is that we have relatively little information on the shape of the value functions and, since we are dealing with short run properties of the equilibrium, we cannot use the per period utility $U\left(x_t, \delta_t\right)$ to approximate the optimal expected utility, as we did in Section 3.

5 Numerical illustration

In this section we illustrate quantitatively the theoretical results of the paper. The analysis of this section shows additional properties of equilibria to complement our theoretical derivations. We also contrast our results with the case when all the information is public. All the details of the numerical algorithm used in this section are in the Appendix.

Assume agents' preferences display constant absolute risk aversion, that is, $u\left(x\right) = -\exp\left(-\rho x\right)$. Section 8.3 of the Appendix shows that with this utility function agents' strategies depend only on their *relative asset position*, that is, on the difference between their holdings of the two assets $x^1 - x^2$ but not on the levels of x^1 and x^2. Then the behavior of an agent in period t depends only on the time period t and on the two individual state variables $x^1 - x^2$ and δ.

Suppose there are two types of agents: half of the agents starts with the initial endowment $(2, 0)$ and the other half starts with the initial endowment $(0, 2)$. For our simulations, we choose a coefficient of absolute risk aversion $\rho = 1$, and the proportion of the informed agents $\alpha = 0.1$. The probability of state S_1 conditional on signal s_1 is equal to $\phi = 0.6$. All the graphs

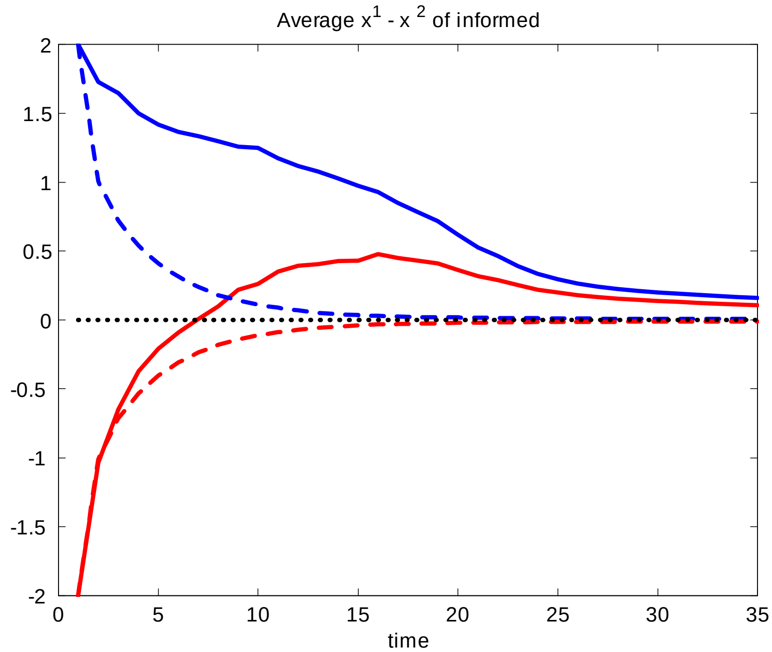

Figure 3: Average relative asset positions of informed agents (conditional on s_1)

depict equilibrium behavior in state s_1, i.e., in the state where asset 1 is more valuable.

Figure 3 shows how the relative asset position $x^1 - x^2$ evolves over time, averaging over all informed agents, both in our model with private information (solid lines) and in the corresponding trading game with complete information (dashed lines).

Consider first the trading game with complete information. The dashed blue line describes the average relative asset position of the rich informed agents, who begin life with larger holdings of asset 1 than of asset 2, i.e., with $x^1_{0,i} - x^2_{0,i} > 0$. Whenever they find other agents willing to trade, these agents sell the first asset and buy the second, until their portfolios are balanced and x^1 is equal to x^2. In this way, the economy converges towards an efficient allocation. The dashed red line describes the behavior of the poor informed agents, who begin life with smaller holdings of asset 1 than of asset 2, i.e. with $x^1_{0,i} - x^2_{0,i} < 0$. In the case of complete information, their behavior is just a mirror image of that of the rich agents.

A different picture arises in the setting with private information. Consider first the rich agents, corresponding to the solid blue line. The rich agents are converging more slowly towards a balanced portfolio for two reasons. First, as illustrated in the static example of Section 4.2, by offering smaller trades they can credibly reveal their information to their opponents and sell asset 1 at more favorable prices. However, a second effect is also at work in a fully dynamic environment: now informed agents have an incentive to hold on to their endowment of asset 1 and wait until the information spreads in the economy. As time passes, more uninformed agents have updated their belief towards $\delta = 1$ and it is easier for the rich informed agents to sell asset 1 at a higher price.

Consider next the poor informed agents, corresponding to the solid red line. These agents know that in the long run uninformed agents will learn the true state of the world and the terms of trade will turn against them. Therefore, they tend to sell the less valuable asset 2 as fast as possible, taking advantage of the relatively favorable prices which uninformed are willing to accept at this stage. After several periods of trading, they cross the zero line and start accumulating a positive relative asset position. At that point, these agents are engaging in purely "speculative" behavior, accumulating the asset 1 by taking advantage of the uninformed agents who are still willing to sell it at a relatively cheap price, with the expectation of reselling it later at higher prices, once the information about s has spread in the economy. That is, these agents start with $x^1 - x^2 < 0$, overshoot to a positive relative asset position $x^1 - x^2 > 0$, and later on converge back to a balanced portfolio.

Relative to the simple static example of Section 4.2, Figure 3 shows an additional reason why the value of information is positive. In that example, the only advantage of informed agents was their ability to reveal their information to the other agents to obtain a more favorable price. This was a gain only for the rich informed agents. Now, instead, also poor informed agents gain from their informational advantage, by engaging in speculative trading.

We can also use our simulations to analyze how the economy converges towards an efficient allocation. Since the graph in Figure 3 only shows the average holdings for some subsets of agents, to study efficiency it is more informative to look at the standard deviation of these holdings across the population. Given that ex post efficiency requires $x^1 - x^2 = 0$ agent by agent and the cross sectional average of $x^1 - x^2$ is zero, the cross sectional standard deviation of $x^1 - x^2$ is a measure of how far the economy is from an efficient allocation. Figure 4 shows the dynamics of this standard deviation. Again, we plot both the dynamics in the case of private information (solid line) and in the case of complete information (dotted line). Consistently with our long run result in Theorem 1, all agents eventually converge to fully balanced portfolios and the economy converges to an efficient allocation. However, the numerical results show that it takes longer to achieve efficiency in the economy with private information. For example, in period $t = 15$ essentially all agents have balanced portfolios in the case of complete information, while agents are still far from an efficient allocation in the private information case.

Figures 5 describes trading volumes, which we define as the average size of trades in the economy (where the size of each trade is measured as the norm of the trade vector). The figure only depicts the case of private information since volumes under public information are virtually identical. One could have conjectured that trading volumes will be higher in this environment as poor informed agents engage in speculative trading in the short run, buying asset 1 only to resell it later. This turns out not to be the case and trading volumes are very similar in the two cases. The reason for this result is that the higher trading volumes generated

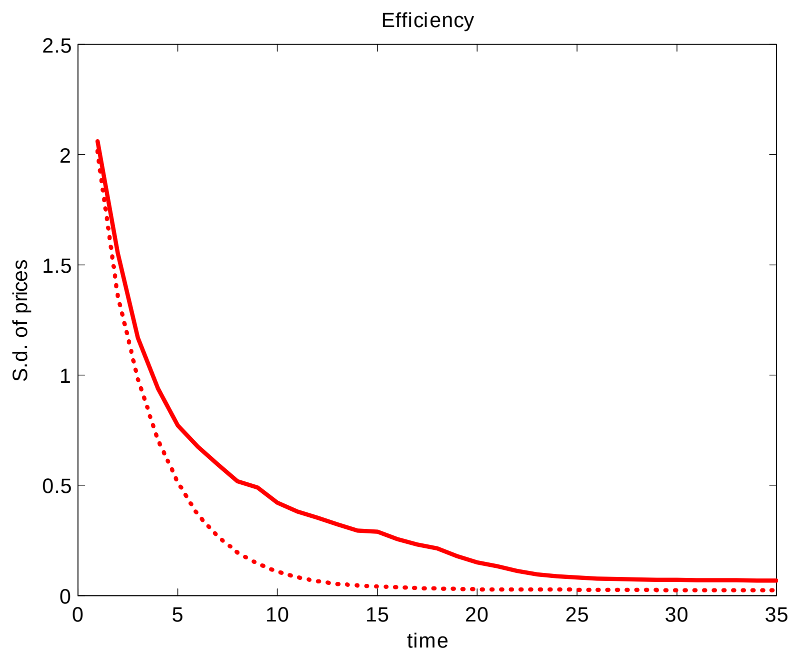

Figure 4: Dispersion of relative asset positions

by the poor informed agents is compensated by the lower trading volumes generated by the rich informed agents who, as we saw above, tend to hold on to asset 1 for a longer time.

Finally, let us look at price dynamics. Given a trade z, the price of asset 2 in terms of asset 1 is equal to the ratio $|z_1| / |z_2|$. The average price across all transactions is plotted in Figure 6 and the standard deviation of prices is plotted in Figure 7. Recall that both paths are conditional on signal s_1. The average price gradually converges to $(1 - \phi) / \phi = 2/3$, which is the fully revealing price in the Walrasian benchmark. Price dispersion is very large at the beginning and, after an initial spike, it decreases gradually. It is interesting to notice that around period $t = 15$ two things happen at the same time: the average price essentially converges to its fully revealing level (Figure 6) and poor informed agents start rebalancing their portfolios (Figure 3). So, in a sense, the "speculative phase" is over in period $t = 15$. From then on, there is still price dispersion, reflecting the fact that agents with different portfolios have different marginal rates of substitution. But the price signals sent by informed agents are now more informative, uninformed agents learn faster, and both quantities and prices converge faster to their long run levels (Figures 4 and 7).

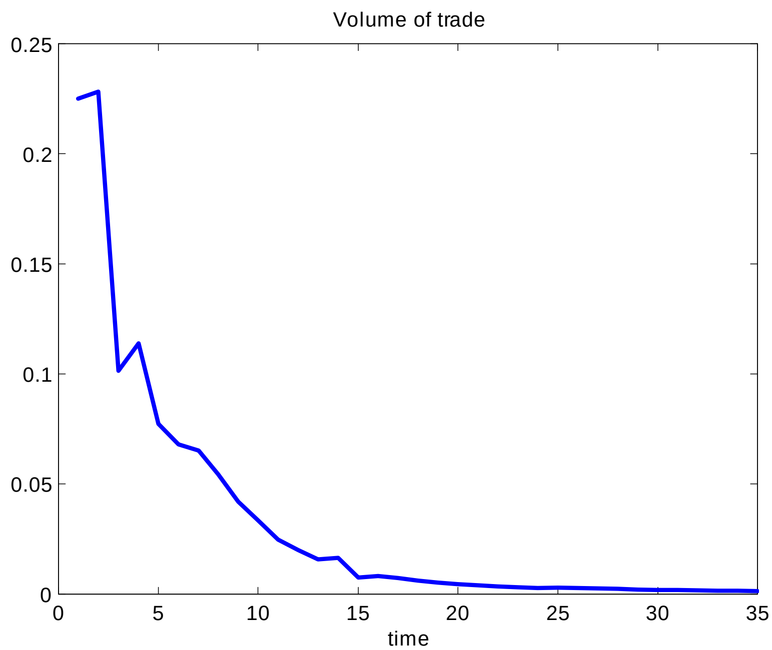

Figure 5: Volume of trade

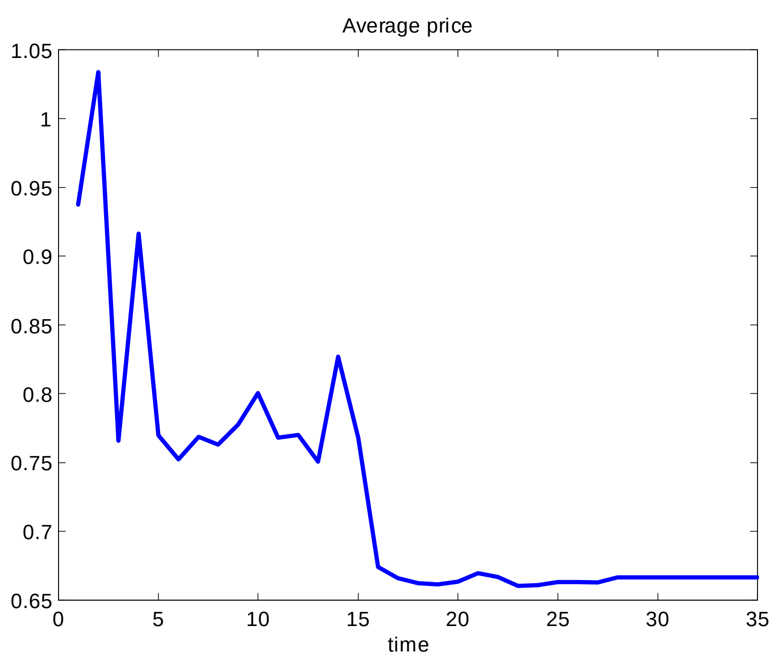

Figure 6: Average path of the price of asset 2 (conditional on s_1)

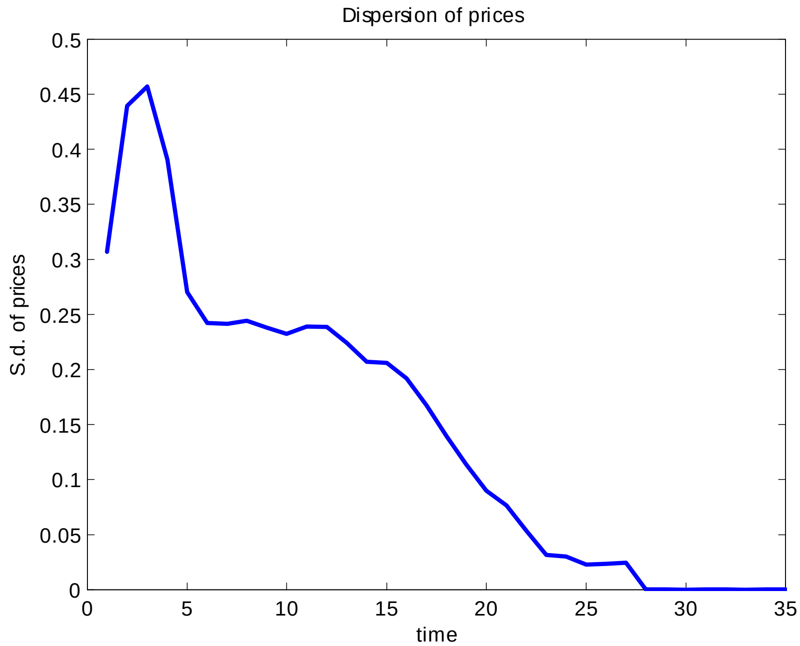

Figure 7: Price dispersion

6 Conclusion

We provide a theory of asset pricing in an environment characterized by two frictions: private information and decentralized trade. These frictions often go hand in hand—it is reasonable to think that in decentralized markets, such as, for example, over the counter markets, agents also receive different pieces of information on the value of the assets traded. We provide theoretical results on convergence to the efficient allocation, on learning by uninformed agents and on the value of information. We also provide a static example and numerical simulations that illustrate and extend the theoretical results.

Throughout the paper we have kept fixed the central trading friction in the model, the frequency of trading captured by the parameter γ. This parameter determines the average number of trades before the game ends. A question that we leave open for future research is what happens in the limit when the trading friction vanishes—i.e., when γ goes to 1—and, in particular, whether the value of information goes to zero.

7 Appendix

7.1 Preliminary results

Here, we introduce two technical results that will be useful throughout the appendix.

Lemma 3 is an elementary probability result (stated without proof) that will be useful whenever we need to establish joint convergence in probability for two or more events.

Lemma 3 *Take two sets $A, B \subset \Omega$ such that $P(A|s) \geq 1 - \varepsilon$ and $P(B|s) > 1 - \eta$ for some positive scalars ε and η. Then, $P(A \cap B|s) > 1 - \varepsilon - \eta$.*

Lemma 4 shows that the portfolios x_t converge to a compact set X in the interior of R^2_+ with probability arbitrarily close to one. This type of set will be used to ensure that several optimization problems used in the proofs are well defined.

Lemma 4 *For any $\varepsilon > 0$ and any state s, there are a compact set $X \subset R^2_{++}$ and a time T such that $P(x_t \in X \mid s) \geq 1 - \varepsilon$ for all $t \geq T$.*

Proof. To prove the lemma, we will find two scalars $\overline{x}$ and $\underline{u}$ such that the set

$$X = \left\{x : x \in (0, \overline{x}]^2, U(x, \delta) \geq \underline{u} \text{ for some } \delta \in [0, 1]\right\}$$

satisfies the desired properties. The proof combines two ideas: use market clearing to put an upper bound on the holdings of the two assets, that is, to show that with probability close to 1 agents have portfolios in $(0, \overline{x}]^2$; use optimality to bound their holdings away from zero, by imposing the inequality $U(x, \delta) \geq \underline{u}$.

First, let us prove that X is a compact subset of R^2_{++}. The following two equalities follow from the fact that $U(x, \delta)$ is continuous, non-decreasing in δ if $x^1 \geq x^2$, and non-increasing if $x^1 \leq x^2$:

$$\begin{aligned}
\left\{x : U(x, \delta) \geq \underline{u} \text{ for some } \delta \in [0, 1], x^1 \geq x^2\right\} &= \left\{x : U(x, 1) \geq \underline{u}, x^1 \geq x^2\right\}, \\
\left\{x : U(x, \delta) \geq \underline{u} \text{ for some } \delta \in [0, 1], x^1 \leq x^2\right\} &= \left\{x : U(x, 0) \geq \underline{u}, x^1 \leq x^2\right\}.
\end{aligned}$$

The sets on the right-hand sides of these equalities are closed sets. Then X can be written as the union of two closed sets, intersected with a bounded set:

$$X = \left(\left\{x : U(x, 1) \geq \underline{u}, x^1 \geq x^2\right\} \cup \left\{x : U(x, 0) \geq \underline{u}, x^1 \leq x^2\right\}\right) \cap (0, \overline{x}]^2,$$

and thus is compact. Notice that $x \notin X$ if $x^j = 0$ for some j because of Assumption 2 and $\underline{u} > -\infty$. Therefore, X is a compact subset of R^2_{++}.

Next, let us define $\overline{x}$ and $\underline{u}$ and the time period T. Given any $\varepsilon > 0$, set $\overline{x} = 4/\varepsilon$. Goods market clearing implies that

$$P(x_t^j > \overline{x}|s) \leq \varepsilon/4 \text{ for all } t, \text{ for } j = 1, 2. \tag{22}$$

To prove this, notice that

$$1 = \int x_t^j(\omega)\, dP(\omega|s) \geq \int_{x_t^j(\omega) > 4/\varepsilon} x_t^j(\omega)\, dP(\omega|s) \geq (4/\varepsilon)\, P(x_t^j > 4/\varepsilon|s),$$

which gives the desired inequality. Let $\overline{u}$ be an upper bound for the agents' utility function $u(\cdot)$ (from Assumption 2). Choose a scalar $u < \overline{u}$ such that

$$\frac{\overline{u} - U(x_0, \delta_0)}{\overline{u} - u} \leq \frac{\varepsilon}{8},$$

for all initial endowments x_0 and initial beliefs δ_0. Such a u exists because $U(x_0, \delta_0) > -\infty$, as initial endowments are strictly positive by Assumption 3, and there is a finite number of types. Then notice that $U(x_0, \delta_0) \leq E\left[v_t|h^0\right]$ for all initial histories h^0, because an agent always has the option to refuse any trade. Moreover

$$E\left[v_t \mid h^0\right] \leq P\left(v_t < u \mid h^0\right) u + P\left(v_t \geq u \mid h^0\right) \overline{u}.$$

Combining these inequalities and rearranging gives

$$P\left(v_t < u \mid h^0\right) \leq \frac{\overline{u} - U(x_0, \delta_0)}{\overline{u} - u} \leq \frac{\varepsilon}{8}.$$

Taking unconditional expectations shows that $P(v_t < u) \leq \varepsilon/8$. Since $P(s) = 1/2$ it follows that

$$P(v_t < u \mid s) \leq \varepsilon/4 \text{ for all } t, \text{ for all } s. \tag{23}$$

Choose T so that

$$P(|u_t - v_t| > u/2 \mid s) \leq \varepsilon/4 \text{ for all } t \geq T. \tag{24}$$

This can be done by Lemma 2, given that almost sure convergence implies convergence in probability. We can then set $\underline{u} = u/2$.

Finally, we check that $P(x_t \in X \mid s) \geq 1 - \varepsilon$ for all $t \geq T$, using the following chain of inequalities:

$$P(x_t \in X \mid s) \geq P\left(x_t \in (0,\overline{x}]^2, U(x_t,\delta_t) \geq \underline{u} \mid s\right) \geq$$
$$P\left(x_t \in (0,\overline{x}]^2, v_t \geq u, |u_t - v_t| \leq u/2 \mid s\right) \geq$$
$$1 - \sum_j P\left(x_t^j > \overline{x} \mid s\right) - P(v_t < u \mid s) - P(|u_t - v_t| > u/2 \mid s) \geq 1 - \varepsilon.$$

The first inequality follows because $U(x_t(\omega), \delta_t(\omega)) \geq \underline{u}$ implies $U(x_t(\omega), \delta) \geq \underline{u}$ for some $\delta \in [0,1]$. The second follows because $v_t(\omega) \geq u$ and $|u_t(\omega) - v_t(\omega)| \leq u/2$ imply $u_t(\omega) = U(x_t(\omega), \delta_t(\omega)) \geq u/2 = \underline{u}$. The third follows from repeatedly applying Lemma 3. The fourth combines (22), (23), and (24). ■

7.2 Proof of Proposition 1

We start by proving a lemma which shows that given any two agents with portfolios in some compact set X, whose marginal rates of substitution differ by at least ε, there is a trade z that achieves a gain in current utility of at least Δ, for some positive Δ.

The lemma is stated in a more general form than what is required to prove Proposition 1. The generality is threefold. First, it applies not just to informed agents but also to agents with any (possibly different) beliefs. Second, we show that the utility gain Δ can be achieved with small trades, i.e., trades such that $\|z\| < \theta$ for some $\theta > 0$. (Throughout the paper, $\|.\|$ is the Euclidean norm). Finally, since an uninformed proposer making offer z can change his beliefs depending on whether his offer is accepted or rejected, we bound the potential utility losses of the proposer under all possible beliefs. These extensions will be useful for later results when we analyze the behavior of uninformed agents.

It will be useful for the rest of the appendix to define the function

$$\mathcal{M}(x,\delta) \equiv \frac{\pi(\delta)\, u'(x^1)}{(1-\pi(\delta))u'(x^2)},$$

which gives the *ex ante* marginal rate of substitution between the two assets for an agent with the portfolio x and belief δ.

Lemma 5 *Let X be a compact subset of R^2_{++}. For any $\varepsilon > 0$ and $\theta > 0$ there is a minimal utility gain $\Delta > 0$ and an amount of asset 1 traded, $\zeta > 0$, with the following property. Take any two agents with portfolios $x_A, x_B \in X$ and beliefs $\delta_A, \delta_B \in [0,1]$ with marginal rates of substitution that differ by more than ε, $\mathcal{M}(x_B,\delta_B) - \mathcal{M}(x_A,\delta_A) > \varepsilon$. Choose any price sufficiently close to the middle of the interval between the two marginal rates of substitution:*

$$p \in [\mathcal{M}(x_A,\delta_A) + \varepsilon/2, \mathcal{M}(x_B,\delta_B) - \varepsilon/2].$$

Then the trade $z = (\zeta, -p\zeta)$ is sufficiently small, $\|z\| < \theta$, and the gain in current utility associated with the trade is larger than or equal to Δ:

$$U(x_A - z, \delta_A) - U(x_A, \delta_A) \geq \Delta, \tag{25}$$

$$U(x_B + z, \delta_B) - U(x_B, \delta_B) \geq \Delta. \tag{26}$$

Moreover, there is a constant $\lambda > 0$, which depends on the set X and on the difference between the marginal rates of substitution ε, but not on the size of the trade θ, such that the potential loss in current utility associated with the trade z is bounded below by $-\lambda\Delta$ for all beliefs δ:

$$U(x_A - z, \delta) - U(x_A, \delta) \geq -\lambda\Delta \textit{ for all } \delta \in [0, 1]. \tag{27}$$

Proof. The idea of the proof is as follows. We construct a Taylor expansion to compute the utility gains for any trade. Then we define the traded amount ζ and the utility gain Δ satisfying (25) and (26).

Choose any two portfolios $x_A, x_B \in X$ and any two beliefs $\delta_A, \delta_B \in [0, 1]$ such that $\mathcal{M}(x_B, \delta_B) - \mathcal{M}(x_A, \delta_A) > \varepsilon$. Pick a price p sufficiently close to the middle of the interval between the marginal rates of substitution:

$$p \in [\mathcal{M}(x_A, \delta_A) + \varepsilon/2, \mathcal{M}(x_B, \delta_B) - \varepsilon/2].$$

This price is chosen so that both agents will make positive gains. Consider agent A and a traded amount $\tilde{\zeta} \leq \bar{\zeta}$ (for some $\bar{\zeta}$ which we will properly choose below). The current utility gain associated with the trade $\tilde{z} = (\tilde{\zeta}, -p\tilde{\zeta})$ can be written as a Taylor expansion:

$$\begin{aligned}
&U(x_A - \tilde{z}, \delta_A) - U(x_A, \delta_A) \\
= \;& -\pi(\delta_A)u'(x_A^1)\tilde{\zeta} + (1 - \pi(\delta_A))u'(x_A^2)p\tilde{\zeta} + \frac{1}{2}\left(\pi(\delta_A)u''(y^1) + (1 - \pi(\delta_A))u''(y^2)p^2\right)\tilde{\zeta}^2 \\
\geq \;& (1 - \pi(\delta_A))u'(x_A^2)(\varepsilon/2)\tilde{\zeta} + \frac{1}{2}\left[\pi(\delta_A)u''(y^1) + (1 - \pi(\delta_A))u''(y^2)p^2\right]\tilde{\zeta}^2,
\end{aligned} \tag{28}$$

for some $(y^1, y^2) \in [x_A^1, x_A^1 - \bar{\zeta}] \times [x_A^2 + p\bar{\zeta}, x_A^2]$. The inequality above follows because $p \geq \mathcal{M}(x_A, \delta_A) + \varepsilon/2$. An analogous expansion can be done for agent B.

Now we want to bound the last line in (28). To do so we first define the minimal and the maximal prices for agents with any belief in $[0, 1]$ and any portfolio in X:

$$\begin{aligned}
\underline{p} &= \min_{x \in X, \delta \in [0,1]} \{\mathcal{M}(x, \delta) + \varepsilon/2\}, \\
\overline{p} &= \max_{x \in X, \delta \in [0,1]} \{\mathcal{M}(x, \delta) - \varepsilon/2\}.
\end{aligned}$$

These prices are well-defined as X is a compact subset of R_{++}^2 and $u(.)$ has continuous first derivative on R^2_{++}. Then, choose $\bar{\zeta} > 0$ such that for all $\tilde{\zeta} \leq \bar{\zeta}$ and all $p \in [\underline{p}, \overline{p}]$, the trade $\tilde{z} = (\tilde{\zeta}, -p\tilde{\zeta})$ satisfies $\|\tilde{z}\| < \theta$ and $x + \tilde{z}$ and $x - \tilde{z}$ are in R^2_+ for all $x \in X$. This means that the trade is small enough. Next, we separately bound from below the two terms in the last line of the Taylor expansion (28). Let

$$\begin{aligned} D'_A &= \min_{x\in X, \delta\in[0,1]} (1-\pi(\delta))u'(x^2)\varepsilon/2, \\ D''_A &= \min_{\substack{x\in X,\delta\in[0,1],\tilde{p}\in[\underline{p},\overline{p}], \\ y\in\left[x^1, x^1+\bar{\zeta}\right]\times[x^2-\tilde{p}\bar{\zeta}, x^2]}} \frac{1}{2}\left[\pi(\delta)u''(y^1) + (1-\pi(\delta))u''(y^2)\tilde{p}^2\right]. \end{aligned}$$

Note that D'_A is positive, D''_A is negative but $D''_A\tilde{\zeta}^2$ is of second order. Then, there exist some $\zeta_A \in \left(0, \bar{\zeta}\right)$ such that, for all $\tilde{\zeta} \leq \zeta_A$,

$$D'_A\tilde{\zeta} + D''_A\tilde{\zeta}^2 > 0$$

and, by construction,

$$U(x_A - \tilde{z}, \delta_A) - U(x_A, \delta_A) \geq D'_A\tilde{\zeta} + D''_A\tilde{\zeta}^2.$$

Analogously, we can find D'_B, D''_B, and ζ_B such that for all $\tilde{\zeta} \leq \zeta_B$ the utility gain for agent B is bounded from below:

$$U(x_B + \tilde{z}, \delta_B) - U(x_A, \delta_B) \geq D'_B\tilde{\zeta} + D''_B\tilde{\zeta}^2 > 0.$$

We are finally ready to define ζ and Δ. Let $\zeta = \min\{\zeta_A, \zeta_B\}$ and

$$\Delta = \min\left\{D'_A\zeta + D''_A\zeta^2, D'_B\zeta + D''_B\zeta^2\right\}.$$

By construction Δ and ζ satisfy the inequalities (25) and (26).

To prove the last part of the lemma, let

$$\lambda = \frac{1}{2}\frac{\pi(1)\min_{x\in X}\left\{u'(x^1)\right\}}{\min\left\{D'_A, D'_B\right\}},$$

which, as stated in the lemma, only depends on X and ε. Using a second-order expansion similar to the one above, the utility gain associated to $z = (\zeta, -p\zeta)$ for an agent with portfolio x_A and any belief $\delta \in [0,1]$, can be bounded below:

$$U(x_A - \tilde{z}, \delta) - U(x_A, \delta) \geq -\pi(1)\min_{x\in X}\left\{u'(x^1)\right\}\zeta + D''_A\zeta^2.$$

Therefore, to ensure that (27) is satisfied, we need to slightly modify the construction above, by choosing ζ so that the following holds

$$\frac{-\pi(1)\min_{x\in X}\left\{u'(x^1)\right\}\zeta + D''_A\zeta^2}{\Delta} > \lambda.$$

The definitions of Δ and λ and a continuity argument show that this inequality holds for some positive $\zeta \leq \min\{\zeta_A, \zeta_B\}$, completing the proof. ■

Proof of Proposition 1. Proceeding by contradiction suppose (6) does not hold. Without loss of generality, let us focus on state s_1. If (6) is violated in s_1 then there exist an $\varepsilon > 0$ and an $\eta \in (0,1)$ such that the following holds for infinitely many periods t:

$$P\left(\left|\mathcal{M}\left(x_t, \delta_t\right) - \kappa\right| > \varepsilon, \delta_t = 1 \mid s_1\right) > \eta P\left(\delta_t = 1 \mid s_1\right) \text{ for all } \kappa. \tag{29}$$

Equation (29) means that the distribution of the informed agents' marginal rates of substitution is such that there is a sufficiently large mass (more than η) which is sufficiently far (more than ε) from any possible value κ.

We want to show that (29) implies that there is a profitable deviation for informed agents. The deviation takes the following form. The informed agent starts deviating at some date T (to be defined) if three conditions are satisfied:

(a) his marginal rate of substitution is below some level κ^* (to be defined): $\mathcal{M}\left(x_T, \delta_T\right) < \kappa^*$;

(b) his utility is close enough to its long run level: $u_T \geq \hat{v}_T - \alpha\eta\Delta/4$ (for some $\Delta > 0$ to be defined);

(c) his portfolio x_T is in some compact set X (to be defined).

We will show that when (a)-(c) hold the agent can make an offer z^* which is accepted with probability $\chi_T\left(z^*|s_1\right) \geq \alpha\eta/4$ and gives him a utility gain of at least Δ. The expected payoff of this strategy at time T is

$$u_T + \chi_T\left(z^*|s_1\right)\left(U(x_T - z^*, \delta_T) - u_T\right) > u_T + \alpha\eta\Delta/4 \geq \hat{v}_T.$$

Since $\hat{v}_T$ is, by definition, the expected payoff of a proposer who follows an optimal strategy, this leads to a contradiction.

To complete the proof we need to define the scalars κ^* and Δ, the set X, the deviating period T and the offer z^*. In the process, we will check that conditions (a)-(c) are satisfied with positive probability, that offer z^* gives a utility gain of at least Δ to the agents who satisfy (a)-(c), and that offer z^* is accepted with probability $\chi_T\left(z^*|s_1\right) \geq \alpha\eta/2$.

Define X to be a compact subset of R^2_{++} such that the portfolios of sufficiently many agents are eventually in this set, i.e., for some T' we have $P\left(x_t \in X \mid s_1\right) \geq 1 - \alpha\eta/4$ for all $t \geq T'$. Such a set exists by Lemma 4.

Define $\Delta > 0$ to be the minimal gain from trade for two agents with marginal rates of substitution that differ by at least ε with portfolios in X. Such a Δ exists by Lemma 5.

We can now find a time T large enough that condition (29) also holds if we restrict attention to agents close to their long run utility, with portfolios in X, i.e., agents who satisfy (b)-(c). Applying Lemmas 2 and 3, choose a $T'' \geq T'$ such that

$$P\left(u_t \geq \hat{v}_t - \alpha\eta\Delta/4, x_t \in X \mid s_1\right) > 1 - \alpha\eta/2 \text{ for all } t \geq T''.$$

Then, using (29) and the fact that there is at least a mass α of informed agents in each period t, we can find a $T \geq T''$ such that there are enough agents (at least $\alpha\eta$) whose marginal rates of substitution are far (at least ε) from any κ:

$$P\left(\left|\mathcal{M}\left(x_T, \delta_T\right) - \kappa\right| > \varepsilon, \delta_T = 1 \mid s_1\right) > \eta P\left(\delta_t = 1 \mid s_1\right) \geq \alpha\eta \text{ for all } \kappa.$$

Using Lemma 3, it follows that at time T there are at least $\alpha\eta/2$ informed agents who satisfy $\left|\mathcal{M}\left(x_T, \delta_T\right) - \kappa\right| > \varepsilon$ for any κ *and* conditions (b)-(c):

$$P\left(\left|\mathcal{M}\left(x_T, \delta_T\right) - \kappa\right| > \varepsilon, u_T \geq \hat{v}_T - \alpha\eta\Delta/4, x_T \in X, \delta_T = 1 \mid s_1\right) > \alpha\eta/2 \text{ for all } \kappa.$$

It will be useful to rewrite this equation as

$$\begin{aligned} & P\left(\left|\mathcal{M}\left(x_T, \delta_T\right) - \kappa\right| \leq \varepsilon, u_T \geq \hat{v}_T - \alpha\eta\Delta/4, x_T \in X, \delta_T = 1 \mid s_1\right) < \\ & \qquad < P\left(u_T \geq \hat{v}_T - \alpha\eta\Delta/4, x_T \in X, \delta_T = 1 \mid s_1\right) - \alpha\eta/2 \text{ for all } \kappa, \end{aligned} \tag{30}$$

To define κ^*, the idea is to use condition (29)—which states that there are not too many informed agents around any κ—to find a κ^* such that enough agents have marginal rate of substitution below κ^* and enough agents have marginal rate of substitution above $\kappa^* + \varepsilon$. The first group of agents will make the offer z^*, the second group will accept it. Define

$$\kappa^* = \sup\left\{\kappa : P\left(\mathcal{M}\left(x_T, \delta_T\right) > \kappa + (3/2)\varepsilon, u_T \geq \hat{v}_T - \alpha\eta\Delta/4, x_T \in X, \delta_T = 1 \mid s_1\right) \geq \alpha\eta/4\right\}.$$

The definition of κ^* implies that there are less than $\alpha\eta/4$ informed agents with marginal rate of substitution above $\kappa^* + 2\varepsilon$ who satisfy (b)-(c),

$$P\left(\mathcal{M}\left(x_T, \delta_T\right) > \kappa^* + 2\varepsilon, u_T \geq \hat{v}_T - \alpha\eta\Delta/4, x_T \in X, \delta_T = 1 \mid s_1\right) < \alpha\eta/4, \tag{31}$$

given that $\kappa^* + \varepsilon/2 > \kappa^*$. Consider the following chain of equalities and inequalities:

$$
\begin{aligned}
&P\left(\mathcal{M}\left(x_T,\delta_T\right) \geq \kappa^*, u_T \geq \hat{v}_T - \alpha\eta\Delta/4, x_T \in X, \delta_T = 1 \mid s_1\right) = \\
&P\left(\kappa^* \leq \mathcal{M}\left(x_T,\delta_T\right) \leq \kappa^* + 2\varepsilon, u_T \geq \hat{v}_T - \alpha\eta\Delta/4, x_T \in X, \delta_T = 1 \mid s_1\right) \\
&P\left(\mathcal{M}\left(x_T,\delta_T\right) > \kappa^* + 2\varepsilon, u_T \geq \hat{v}_T - \alpha\eta\Delta/4, x_T \in X, \delta_T = 1 \mid s_1\right) < \\
&\quad < P\left(u_T \geq \hat{v}_T - \alpha\eta\Delta/4, x_T \in X, \delta_T = 1 \mid s_1\right) - \alpha\eta/4,
\end{aligned}
$$

where the equalities are immediate and the inequality follows from (30) (with $\kappa = \kappa^* + \varepsilon$) and (31). This implies

$$
P\left(\mathcal{M}\left(x_T,\delta_T\right) < \kappa^*, u_T \geq \hat{v}_T - \alpha\eta\Delta/4, x_T \in X, \delta_T = 1 \mid s_1\right) > 0, \tag{32}
$$

which shows that conditions (a)-(c) are met with positive probability.

To define the deviating offer z^*, notice that, by the definition of Δ, there exists an offer $z^* = (\zeta^*, -p^*\zeta^*)$, with price $p^* = \kappa^* + \varepsilon/2$, such that

$$
U(x - z^*, \delta) \geq U(x,\delta) + \Delta \text{ if } \mathcal{M}\left(x,\delta\right) < \kappa^* \text{ and } x \in X, \tag{33}
$$

$$
U(x + z^*, \delta) \geq U(x,\delta) + \Delta \text{ if } \mathcal{M}\left(x,\delta\right) > \kappa^* + \varepsilon \text{ and } x \in X. \tag{34}
$$

Condition (33) shows that an informed proposer who satisfies (a)-(c) gains at least Δ if offer z^* is accepted.

Finally, the definition of κ^* implies that there must be at least $\alpha\eta/4$ agents with marginal rate of substitution above $\kappa^* + \varepsilon$,

$$
P\left(\mathcal{M}\left(x_T,\delta_T\right) > \kappa^* + \varepsilon, u_T \geq \hat{v}_T - \alpha\eta\Delta/2, x_T \in X, \delta_T = 1 \mid s_1\right) \geq \alpha\eta/4, \tag{35}
$$

given that $\kappa^* - \varepsilon/2 < \kappa^*$. Recall that $\hat{v}_t$ represents, by definition, the maximal expected utility the responder can get from rejecting all offers and behaving optimally in the future. A responder who receives z^* has the option to accept it and stop trading from then on, which yields expected utility $U(x_T + z^*, \delta_T)$. For all informed agents who satisfy $\mathcal{M}\left(x_T,\delta_T\right) \geq \kappa^* + \varepsilon$, $u_T \geq \hat{v}_T - \alpha\eta\Delta/4$ and $x_T \in X$, we have the chain of inequalities

$$
U(x_T + z^*, \delta_T) \geq u_T + \Delta > u_T + \alpha\eta\Delta/4 \geq \hat{v}_T,
$$

where the first inequality follows from (34). This shows that rejecting z^* at time T is a strictly dominated strategy for these informed agents. Since there are at least $\alpha\eta/4$ of them, by (35), the probability that z^* is accepted must then satisfy $\chi_T\left(z|s_1\right) \geq \alpha\eta/4$. ■

7.3 Proof of Proposition 2

The proof proceeds by contradiction, supposing that for all $\varepsilon > 0$ there are infinitely many periods t in which: (a) the long-run marginal rates of substitution of informed agents in states s_1 and s_2 are similar, $|\kappa_t(s_1) - \kappa_t(s_2)| < \varepsilon$, and (b) almost all informed and uninformed agents have marginal rates of substitution close to $\kappa_t(s)$, i.e.,

$$P\left(\left|\mathcal{M}\left(x_t, \delta_t\right) - \kappa_t(s)\right| < 2\varepsilon \mid s\right) > 1 - \varepsilon$$

in some state s.

First of all, let us manipulate this expression, to derive a version that is easier to contradict. Specifically, let us show that (a) and (b) imply that there are infinitely many periods in which almost all agents have marginal rates of substitution near 1. By symmetry, the long-run marginal rates of substitutions of informed agents in states s_1 and s_2 are one the inverse of the other:

$$\kappa_t(s_1) = 1/\kappa_t(s_2).$$

Then, some algebra shows that $|\kappa_t(s_1) - \kappa_t(s_2)| < \varepsilon$ implies $|\kappa_t(s_1) - 1| < \varepsilon$. Moreover, by the triangle inequality, $|\mathcal{M}(x_t, \delta_t) - \kappa_t(s_1)| < 2\varepsilon$ and $|\kappa_t(s_1) - 1| < \varepsilon$ imply $|\mathcal{M}(x_t, \delta_t) - 1| < 3\varepsilon$. Therefore, (a) and (b) imply that for all $\varepsilon > 0$ there are infinitely many periods t in which almost all agents have marginal rates of substitution close to 1, i.e., $P(|\mathcal{M}(x_t, \delta_t) - 1| < 3\varepsilon \mid s) > 1 - \varepsilon$ in some state s. Without loss of generality, we focus on the case where this condition holds for infinitely many periods in state s_1,

$$P\left(\left|\mathcal{M}\left(x_t, \delta_t\right) - 1\right| < 3\varepsilon \mid s_1\right) > 1 - \varepsilon. \tag{36}$$

The idea of the proof is to show that when almost all agents have marginal rates of substitution close to 1 in state s_1, agents will hold on average more of asset 1 than of asset 2. This contradicts market clearing, which requires the average holdings of the two assets to be equal:

$$\int \left(x_t^1(\omega) - x_t^2(\omega)\right) dP(\omega|s_1) = 0. \tag{37}$$

Informed agents have belief $\delta_t = 1$ in state 1, so if their marginal rates of substitution are close enough to 1 they will clearly have larger holdings of asset 1 than of asset 2. The main difficulty of the proof is to make sure that there aren't too many uninformed agents with larger holdings of asset 2 than of asset 1. Since also uninformed agents have marginal rates of substitution near 1, they can hold more asset 2 than asset 1 only if their beliefs are biased towards state s_2, i.e., if they have $\delta_t < 1/2$. However, we will argue that if the true state is s_1 there are always more uninformed agents biased towards s_1 than uninformed agents biased towards s_2. Therefore, the agents in the economy will hold, on average, more of asset 2 than asset 1. The main formal step for this argument is given by the following lemma. We will complete the proof of Proposition 2 after stating and proving the lemma.

The lemma shows that we can start from the equilibrium distribution of portfolios and beliefs implied by $P(\omega|s_1)$ and construct an auxiliary distribution of portfolios and beliefs, G_t, with the following three properties: (i) it only includes agents with beliefs greater than 1/2; (ii) it includes the same mass of informed agents as the original distribution; (iii) the average holdings of assets 1 and 2 are equalized. In particular, G_t is constructed by eliminating symmetric masses of agents with $\delta_t > 1/2$ and $\delta_t < 1/2$ and with symmetric holdings of the two assets: if the agents in the first group hold (x^1, x^2) and have belief δ, we find a group of agents with holdings (x^2, x^1) and belief $1-\delta$, and reduce the masses of both groups by an equal amount. Since $(x^1 - x^2) = -(x^2 - x^1)$, this procedure ensures that the average holdings of the two assets are still equalized under the new measure. Moreover, Bayesian reasoning implies that the first group is always larger than the second, so we can construct G_t leaving only a positive mass of agents in the first group. By this process, we end up with a distribution where every agent has $\delta_t(\omega) > 1/2$ and the average portfolios of goods 1 and 2 are equal.

Notice that the lemma is stated using a modified version of condition (37). That is, instead of showing the equality of average asset holdings of assets 1 and 2, we truncate the portfolio distribution, imposing $x_t^2 \leq m$ for some arbitrarily large m, and show that the average holdings of asset 1 can only exceed the average holdings of asset 2 by an arbitrarily small $\varepsilon > 0$. Here is where we exploit the assumption of uniform market clearing. This property will be useful when completing the proof of Proposition 2.

Lemma 6 *For all $\varepsilon > 0$, there are a scalar M and a sequence of (discrete) measures G_t on the space of portfolios and beliefs $R_+^2 \times [0,1]$ such that the following properties are satisfied: (i) the measure is zero for all beliefs smaller than or equal to $1/2$:*

$$G_t(x,\delta) = 0 \text{ if } \delta \leq 1/2;$$

(ii) G_t corresponds to the distribution generated by the measure P conditional on s_1 for informed agents:

$$G_t(x,1) = P(\omega : x_t(\omega) = x, \delta_t(\omega) = 1 \mid s_1) \text{ for all } x \text{ and } t;$$

(iii) the average holdings of asset 1 exceed the average holdings of asset 2, truncated at any $m \geq M$, by less than ε:

$$\int_{x^2 \leq m} (x^1 - x^2)\, dG_t(x,\delta) \leq \varepsilon \text{ for all } m \geq M \text{ and all } t. \tag{38}$$

Proof. For all $x \in R^2$ and all $\delta \in [0,1]$ define the measure G_t as follows

$$G_t(x,\delta) \equiv \begin{cases} P(\omega : x_t(\omega) = x, \delta_t(\omega) = \delta \mid s_1) - P(\omega : x_t(\omega) = x, \delta_t(\omega) = \delta \mid s_2) & \text{if } \delta > 1/2 \\ 0 & \text{if } \delta \leq 1/2 \end{cases}.$$

We first prove that G_t is a well defined measure and next we prove properties (i)-(iii).

Since P generates a discrete distribution over x and δ for each t, to prove that G_t is a well defined measure we only need to check that

$$P(x_t = x, \delta_t = \delta \mid s_2) \leq P(x_t = x, \delta_t = \delta \mid s_1)$$

so that G_t is non-negative. Take any $\delta > 1/2$. Bayesian rationality implies that a consumer who knows his belief is δ must assign probability δ to s_1:

$$\delta = P(s_1 \mid x_t = x, \delta_t = \delta).$$

Moreover, Bayes' rule implies that

$$\frac{P(s_2 \mid x_t = x, \delta_t = \delta)}{P(s_1 \mid x_t = x, \delta_t = \delta)} = \frac{P(x_t = x, \delta_t = \delta \mid s_2) P(s_2)}{P(x_t = x, \delta_t = \delta \mid s_1) P(s_1)}.$$

Rearranging and using $P(s_1) = P(s_2)$ and $\delta > 1/2$, yields

$$\frac{P(x_t = x, \delta_t = \delta \mid s_2)}{P(x_t = x, \delta_t = \delta \mid s_1)} = \frac{1-\delta}{\delta} < 1,$$

which gives the desired inequality.

Property (i) is immediately satisfied by construction. Property (ii) follows because $P(x_t = x, \delta_t = 1 \mid s_2) = 0$ for all x, given that $\delta_t = 1$ requires that we are at a history which arises with zero probability conditional on s_2. The proof of property (iii) is longer and involves the manipulation of market clearing relations and the use of our symmetry assumption. Using the assumption of uniform market clearing, find an M such that

$$\int_{x_t^2(\omega) \leq m} x_t^2(\omega)\, dP(\omega|s_1) \geq 1 - \varepsilon \text{ for all } m \geq M. \tag{39}$$

Notice that

$$\int_{x_t^2(\omega) \leq m} x_t^1(\omega)\, dP(\omega|s_1) \leq \int x_t^1(\omega)\, dP(\omega|s_1) = 1.$$

Which combined with (39) implies that

$$\int_{x_t^2(\omega) \leq m} \left(x_t^1(\omega) - x_t^2(\omega)\right) dP(\omega|s_1) \leq \varepsilon \text{ for all } m \geq M.$$

Decomposing the integral on the left-hand side gives

$$\int_{\substack{x_t^1>x_t^2\\x_t\in[0,m]^2}} \left(x_t^1 - x_t^2\right) dP(\omega|s_1) + \int_{\substack{x_t^2=x_t^1\\x_t\in[0,m]^2}} \left(x_t^1 - x_t^2\right) dP(\omega|s_1) + \int_{\substack{x_t^1<x_t^2\\x_t\in[0,m]^2}} \left(x_t^1 - x_t^2\right) dP(\omega|s_1)$$
$$+ \int_{\substack{x_t^1>m\\x_t^2\leq m}} \left(x_t^1 - x_t^2\right) dP(\omega|s_1) \leq \varepsilon. \tag{40}$$

Let us first focus on the first three terms on the left-hand side of this expression. The second term is zero. Using symmetry to replace the third term, the sum of the first three terms can then be rewritten as

$$\int_{\substack{x_t^1>x_t^2\\x_t\in[0,m]^2}} \left(x_t^1 - x_t^2\right) dP(\omega|s_1) + \int_{\substack{x_t^1>x_t^2\\x_t\in[0,m]^2}} \left(x_t^2 - x_t^1\right) dP(\omega|s_2). \tag{41}$$

These two integrals are equal to the sums of a finite number of non-zero terms, one for each value of x and δ with positive mass. Summing the corresponding terms in each integral, we have three cases: (a) terms with $\delta_t = \delta > 1/2$ and $P(x_t = x, \delta_t = \delta|s_1) > P(x_t = x, \delta_t = \delta|s_2)$ (by Bayes' rule), which can be written as

$$\begin{aligned}\left(x^1 - x^2\right) P(x_t &= x, \delta_t = \delta|s_1) - \left(x^1 - x^2\right) P(x_t = x, \delta_t = \delta|s_2)\\ &= \left(x^1 - x^2\right) G_t\left(x\right);\end{aligned}$$

(b) terms with $\delta_t = \delta = 1/2$ and $P(x_t = x, \delta_t = \delta|s_1) = P(x_t = x, \delta_t = \delta|s_2)$ (by Bayes' rule), which are equal to zero,

$$\left(x^1 - x^2\right) P(x_t = x, \delta_t = \delta|s_1) - \left(x^1 - x^2\right) P(x_t = x, \delta_t = \delta|s_2) = 0;$$

(c) terms with $\delta_t = \delta < 1/2$ and $P(x_t = x, \delta_t = \delta|s_1) = P(x_t = x, \delta_t = \delta|s_2)$ (once more, by Bayes' rule), which can be rewritten as follows, exploiting symmetry,

$$\begin{aligned}&\left(x^1 - x^2\right) P\left(x_t = \left(x^1, x^2\right), \delta_t = \delta|s_1\right) - \left(x^1 - x^2\right) P\left(x_t = \left(x^1, x^2\right), \delta_t = \delta|s_2\right)\\ &= \left(x^1 - x^2\right)\left[P\left(x_t = \left(x^2, x^1\right), \delta_t = 1-\delta|s_2\right) - P\left(x_t = \left(x^2, x^1\right), \delta_t = 1-\delta|s_1\right)\right]\\ &= \left(x^2 - x^1\right) G_t\left(\left(x^2, x^1\right), 1-\delta\right).\end{aligned}$$

Combining all these terms, the integral (41) is equal to

$$\int_{\substack{x^1>x^2,\delta>1/2\\ x\in[0,m]^2}} \left(x^1-x^2\right) dG_t\left(x,\delta\right) + \int_{\substack{x^1>x^2,\delta<1/2\\ x\in[0,m]^2}} \left(x^2-x^1\right) dG_t\left(\left(x^2,x^1\right),1-\delta\right)$$
$$= \int_{\substack{x^1>x^2,\delta>1/2\\ x\in[0,m]^2}} \left(x^1-x^2\right) dG_t\left(x,\delta\right) + \int_{\substack{x^2>x^1,\delta>1/2\\ x\in[0,m]^2}} \left(x^1-x^2\right) dG_t\left(x,\delta\right) =$$
$$= \int_{x\in[0,m]^2} \left(x^1-x^2\right) dG_t\left(x,\delta\right),$$

where the first equality follows from a change of variables and the second from the fact that G_t is zero for all $\delta \leq 1/2$. We can now go back to the integral on the right-hand side of (40), and notice that the integrand $\left(x_t^1 - x_t^2\right)$ in the fourth term is positive, so replacing the measure P with the measure G_t, which is smaller or equal than P, reduces the value of that term. Therefore the inequality (40) in terms of the measure P, leads to the following inequality in terms of the measure G_t

$$\int_{x^2\leq m} \left(x^1-x^2\right) dG_t \leq \varepsilon,$$

completing the proof of property (iii). ■

We can now complete the proof of the proposition.

Proof of Proposition 2. We now proceed to use the two conditions (36) and (38) introduced above to reach a contradiction. We first show that agents with marginal rates of substitution near 1 and beliefs greater than 1/2 must hold more of asset 1 than of asset 2. Then we show that such asset holdings violate market clearing.

Formally, our objective is to show that, for some appropriately chosen positive scalars m and ζ, the following inequality holds for some t^*

$$\int_{x^2\leq m} \left(x^1-x^2\right) dG_{t^*} > \zeta, \tag{42}$$

and then showing that this contradicts (38). To evaluate the integral in (42) we will divide the agents into three groups.

Group 1: Informed agents with marginal rate of substitution sufficiently close to 1 and portfolios in some compact set X. We will prove that for these agents the difference $x^1 - x^2$ is bounded below by some positive number.

Group 2: Uninformed agents with marginal rate of substitution sufficiently close to 1 and informed agents with marginal rate of substitution sufficiently close to 1 but portfolios not in X. We will prove that, for all such agents the difference $x^1 - x^2$ is bounded below by some small negative number.

Group 3: Agents with marginal rate of substitution far from 1 at time t. We will prove that the measure of such agents goes to zero.

In the rest of the proof, we construct the three groups above, we define the constants m and ζ, we find period t^*, and, finally, we prove inequality (42).

Step 1 (Group 1). Since there is at least a mass α of informed agents, using Lemmas 3 and 4, we can find a compact set $X \subset R^2_{++}$ and a time T such that for all $\varepsilon > 0$ there is a large enough mass of informed agents with (a) marginal rate of substitution close to 1 (within 3ε) and (b) portfolios in the set X, that is,

$$P\left(\left|\mathcal{M}\left(x_t, \delta_t\right) - 1\right| < 3\varepsilon, \delta_t = 1, x_t \in X \mid s_1\right) > (5/6)\,\alpha - \varepsilon \tag{43}$$

for all periods $t \geq T$ in which $P\left(\left|\mathcal{M}\left(x_t, \delta_t\right) - 1\right| < 3\varepsilon \mid s_1\right) > 1 - \varepsilon$.

Consider the following minimization problem

$$\begin{aligned} d_I\left(\varepsilon\right) \quad = \quad & \min_{x \in X} \left(x^1 - x^2\right) \\ & \text{s.t. } \left|\mathcal{M}\left(x, 1\right) - 1\right| \leq 3\varepsilon. \end{aligned}$$

The value $d_I\left(\varepsilon\right)$ is the minimal difference between the holdings of the two assets, for informed agents who satisfy (a) and (b). For future reference, notice that $d_I\left(\varepsilon\right)$ is continuous, from the theorem of the maximum.

Consider this problem with $\varepsilon = 0$. Let us prove that $d_I\left(0\right) > 0$. If $x^1 \leq x^2$, then $u'\left(x^1\right) \geq u'\left(x^2\right)$ and, therefore, the marginal rate of substitution

$$\mathcal{M}\left(x, 1\right) = \frac{\pi(1) u'(x^1)}{\left(1 - \pi(1)\right) u'(x^2)} \geq \frac{\pi(1)}{1 - \pi(1)} > 1.$$

Therefore, all x that satisfy $\left|\mathcal{M}\left(x, 1\right) - 1\right| \leq 0$ must also satisfy $x^1 > x^2$. In other words, given that informed agents have a signal favorable to state 1, if their marginal rate of substitution is exactly 1 they must hold strictly more of asset 1.

We can now define the constant ζ—the lower bound for the average difference in the holdings of assets 1 and 2 in expression (42)—as

$$\zeta = \frac{\alpha}{6} d_I\left(0\right).$$

Next, we define the quantity m. Applying uniform market clearing and Lemma 6, we can find an $m \geq d_I\left(0\right)$ such that the following inequalities hold for all t:

$$\int_{x_t^2 > m} x_t^2 dP(\omega | s_1) \leq \zeta \tag{44}$$

and

$$\int_{x^2 \leq m} \left(x^1 - x^2\right) dG_t \leq \zeta. \tag{45}$$

From (44), we have

$$mP\left(x_t^2(\omega) > m\right) \leq \int_{x_t^2(\omega) > m} x_t^2(\omega)\, dP(\omega | s_1) \leq \zeta \text{ for all } t,$$

which, given the definition of ζ and the fact that $m \geq d_I(0)$, implies

$$P\left(x_t^2(\omega) > m\right) \leq \frac{\alpha}{6} \frac{d_I(0)}{m} \leq \frac{\alpha}{6} \text{ for all } t.$$

We then obtain the following chain of equalities and inequalities,

$$\begin{aligned}
& P\left(|\mathcal{M}(x_t, \delta_t) - 1| < 3\varepsilon, \delta_t = 1, x_t \in X \mid s_1\right) = \\
& P\left(|\mathcal{M}(x_t, \delta_t) - 1| < 3\varepsilon, \delta_t = 1, x_t^2 \leq m, x_t \in X \mid s_1\right) \\
& \quad + P\left(|\mathcal{M}(x_t, \delta_t) - 1| < 3\varepsilon, \delta_t = 1, x_t^2 > m, x_t \in X \mid s_1\right) \\
\leq \;\; & P\left(|\mathcal{M}(x_t, \delta_t) - 1| < 3\varepsilon, \delta_t = 1, x_t^2 \leq m, x_t \in X \mid s_1\right) + \alpha/6,
\end{aligned}$$

and combine it with (43) to conclude that

$$P\left(|\mathcal{M}(x_t, \delta_t) - 1| < 3\varepsilon, \delta_t = 1, x_t^2 \leq m, x_t \in X \mid s_1\right) > (2/3)\,\alpha - \varepsilon \tag{46}$$

for all $t \geq T$ in which $P(|\mathcal{M}(x_t, \delta_t) - 1| < 3\varepsilon \mid s_1) > 1 - \varepsilon$.

We are almost ready to construct group 1 as the set of informed agents that satisfy (a) and (b) above, plus the additional restriction $x_t^2 \leq m$, for appropriately chosen values of t and ε. The last step of this construction is to choose t and ε, but we will only be able to do so after constructing group 2 in the next step.

Step 2 (Groups 2 and 3). Consider the problem

$$\begin{aligned}
d_U(\varepsilon) \;\; = \;\; & \min_{\substack{x^2 \leq m \\ \delta \geq 1/2}} \left(x^1 - x^2\right) \\
& \text{s.t. } \; |\mathcal{M}(x, \delta) - 1| \leq 3\varepsilon.
\end{aligned}$$

The value $d_U(\varepsilon)$ is the minimum difference between the holdings of the two assets for all agents: (a) with marginal rates of substitution sufficiently close to 1, (b) holdings of asset 2 less than or equal to m, and (c) beliefs above 1/2. The theorem of the maximum implies that $d_U(\varepsilon)$ is continuous. Moreover, $d_U(\varepsilon)$ is negative for all $\varepsilon > 0$ and $d_U(0) = 0$.

Recall from Step 1 that $d_I(\varepsilon)$ is continuous and $d_I(0) > 0$. It is then possible to find a positive ε^*, smaller than both $\alpha/6$ and ζ/m, such that

$$\frac{\alpha}{2}d_I(\varepsilon^*) + d_U(\varepsilon^*) > \frac{\alpha}{3}d_I(0) = 2\zeta, \tag{47}$$

(the second equality comes from the definition of ζ).

Since, by construction $\varepsilon^* < \alpha/6$, it follows from (46) that the mass of informed agents with marginal rates of substitution near 1 (within $3\varepsilon^*$) and a portfolio that satisfies $x_t^2 \leq m$ and $x_t \in X$ is sufficiently high:

$$P\left(|\mathcal{M}(x_t,\delta_t) - 1| < 3\varepsilon^*, \delta_t = 1, x_t^2 \leq m, x_t \in X \mid s_1\right) > \alpha/2 \tag{48}$$

for all $t \geq T$ in which $P(|\mathcal{M}(x_t,\delta_t) - 1| < 3\varepsilon^* \mid s_1) > 1 - \varepsilon^*$.

Moreover, by Lemma 3, in all periods $t \geq T$ in which almost all agents have marginal rate of substitution close to 1, i.e., $P(|\mathcal{M}(x_t,\delta_t) - 1| < 3\varepsilon^* \mid s_1) > 1 - \varepsilon^*$, almost all agents with beliefs higher than 1/2 and portfolios satisfying $x_t^2 \leq m$ also have a marginal rate of substitution close to 1:

$$P\left(|\mathcal{M}(x_t,\delta_t) - 1| < 3\varepsilon^*, \delta_t > 1/2, x_t^2 \leq m \mid s_1\right) > P\left(\delta_t > 1/2, x_t^2 \leq m \mid s_1\right) - \varepsilon^*. \tag{49}$$

By hypothesis, i.e., by (36), we can choose a $t^* \geq T$ such that

$$P(|\mathcal{M}(x_{t^*},\delta_{t^*}) - 1| < 3\varepsilon^* \mid s_1) > 1 - \varepsilon^*$$

so that both (48) and (49) are satisfied.

We can finally define groups 1, 2 and 3 as follows

$$\begin{aligned}
A_1 &= \left\{(x,\delta) : |\mathcal{M}(x,\delta) - 1| < 3\varepsilon^*, \delta = 1, x^2 \leq m, x \in X\right\}, \\
A_2 &= \left\{(x,\delta) \notin A_1 : |\mathcal{M}(x,\delta) - 1| < 3\varepsilon^*, \delta > 1/2, x^2 \leq m\right\}, \\
A_3 &= \left\{(x,\delta) \notin A_1 \cup A_2 : \delta > 1/2, x^2 \leq m\right\}.
\end{aligned}$$

Step 3. Now we split the integral (42) in three parts, corresponding to groups 1, 2, and 3, and determine a lower bound for each of them. First, we have

$$\int_{A_1} \left(x^1 - x^2\right) dG_{t^*} = \int_{(x_{t^*},\delta_{t^*})\in A_1} \left(x_{t^*}^1(\omega) - x_{t^*}^2(\omega)\right) dP(\omega|s_1) \geq \frac{\alpha}{2}d_I(\varepsilon^*), \tag{50}$$

where the equality follows from property (ii) of the distribution G_t (in Lemma 6) and the inequality follows from the definition of $d_I(\varepsilon^*)$ and condition (48). The definition of $d_U(\varepsilon^*)$ implies that

$$\int_{A_2} \left(x^1 - x^2\right) dG_{t^*} \geq d_U(\varepsilon^*) P(A_2) \geq d_U(\varepsilon^*), \tag{51}$$

since $d_U(\varepsilon^*) < 0$ and $P(A_2) \leq 1$. Finally, the definition of the measure G_t and condition (49) imply that

$$G_{t^*}(A_3) \leq P((x_{t^*}, \delta_{t^*}) \in A_3 | s_1) \leq P\left(\delta_{t^*} > 1/2, x_{t^*}^2 \leq m | s_1\right) - P((x_{t^*}, \delta_{t^*}) \in A_1 \cup A_2 | s_1) \leq \varepsilon^* < \zeta/m,$$

where the last inequality follows from the definition of ε^*. We then have the following lower bound

$$\int_{A_3} \left(x^1 - x^2\right) dG_{t^*} \geq -mG_{t^*}(A_3) \geq -\zeta. \tag{52}$$

We can now combine (50), (51) and (52) and use inequality (47) to obtain a lower bound for the whole integral (42):

$$\int_{x^2 \leq m} \left(x^1 - x^2\right) dG_{t^*} \geq \frac{\alpha}{2} d_I(\varepsilon^*) + d_U(\varepsilon^*) - \zeta > \zeta.$$

Comparing this inequality and (45) leads to the desired contradiction. ■

7.4 Proof of Proposition 3

Proof. We start with the usual convergence properties. Since the marginal rates of substitution of informed agents converge, by Proposition 1, and there is at least a mass α of informed agents, using Lemmas 3 and 4 we can find a compact set $X \subset R_{++}^2$ and a time T' such that there is a sufficiently large mass of informed agents with marginal rates of substitution sufficiently close to $\kappa_t(s)$ (within $\bar{\varepsilon}/2$) and portfolios in X:

$$P\left(|\mathcal{M}(x_t, \delta_t) - \kappa_t(s)| < \bar{\varepsilon}/2, \delta_t = \delta^I(s), x_t \in X \mid s\right) > (3/4)\alpha$$

for all $t \geq T'$ and for all s, where $\bar{\varepsilon}$ is defined as in Proposition 2.

Now we provide an important concept. We want to focus on the utility gains that can be achieved by small trades (of norm less than θ), by agents with marginal rates of substitution sufficiently different from each other (by at least $\bar{\varepsilon}/2$). Formally, we proceed as follows. Take any $\theta > 0$. Using Lemma 5, we can then find a lower bound for the utility gain $\Delta > 0$ from trade between two agents with marginal rates of substitution differing by at least $\bar{\varepsilon}/2$ and with portfolios in X, making trades of norm less than θ. It is important to notice that this is the gain achieved if the agents trade but do not change their beliefs. Therefore, it is also important to bound from below the gains that can be achieved by such trades if beliefs are updated in the most pessimistic way. This bound is also given by Lemma 5, which ensures that $-\lambda\Delta$ is a lower bound for the gains of the agent offering z at any possible ex post belief (where λ is a positive scalar independent of θ).

Next we want to restrict attention to agents who are close to their long-run expected utility. Per period utility u_t converges to the long run value $\hat{v}_t$, by Lemma 2. We can then apply Lemma 3 and find a time period $T \geq T'$ such that, for all $t \geq T$ and for all s:

$$P\left(u_t \geq \hat{v}_t - \alpha\Delta/4, x_t \in X \mid s\right) > 1 - \bar{\varepsilon}/2, \tag{53}$$

and

$$P\left(\left|\mathcal{M}\left(x_t, \delta_t\right) - \kappa_t\left(s\right)\right| < \bar{\varepsilon}/4, u_t \geq \hat{v}_t - \alpha\Delta/8, \delta_t = \delta^I\left(s\right), x_t \in X \mid s\right) > \alpha/2. \tag{54}$$

Equation (53) states that there are enough agents, both informed and uninformed, close to their long run utility. Equation (53) states that there are enough informed agents close to both their long run utility and to their long-run marginal rates of substitution.

We are now done with the preliminary steps ensuring proper convergence and can proceed to the body of the argument.

Choose any $t \geq T$. By Proposition 2, two cases are possible: (i) either the informed agents' long run marginal rates of substitution are far enough from each other, $|\kappa_t(s_1) - \kappa_t(s_2)| \geq \bar{\varepsilon}$; or (ii) they are close to each other, $|\kappa_t(s_1) - \kappa_t(s_2)| < \bar{\varepsilon}$, but there is a large enough mass of uninformed agents with marginal rate of substitution far from that of the informed agents, $P\left(\left|\mathcal{M}\left(x_t, \delta_t\right) - \kappa_t(s)\right| \geq 2\bar{\varepsilon} \mid s\right) \geq \bar{\varepsilon}$ for all s.

In the next two steps, we construct the desired trade z for each of these two cases, and then complete the argument in step 3.

Step 1. Consider the first case, in which $|\kappa_t(s_1) - \kappa_t(s_2)| \geq \bar{\varepsilon}$. In this case, an uninformed agent can exploit the difference between the informed agents' marginal rates of substitution in states s_1 and s_2, making an offer at an intermediate price. This offer will be accepted with higher probability in the state in which the informed agents' marginal rate of substitution is higher. In particular, suppose

$$\kappa_t\left(s_2\right) + \bar{\varepsilon} \leq \kappa_t\left(s_1\right)$$

(the opposite case is treated symmetrically). Lemma 5 and the definition of the utility gain Δ imply that there is a trade $z = (\zeta, -p\zeta)$, with price $p = (\kappa_t\left(s_1\right) + \kappa_t\left(s_2\right))/2$ and size $\|z\| < \theta$, that satisfies the following inequalities:

$$U(x_t + z, \delta_t) \geq u_t + \Delta \text{ if } \mathcal{M}\left(x_t, \delta_t\right) > \kappa_t\left(s_1\right) - \bar{\varepsilon}/4 \text{ and } x_t \in X, \tag{55}$$

$$U(x_t - z, \delta_t) \geq u_t + \Delta \text{ if } \mathcal{M}\left(x_t, \delta_t\right) < \kappa_t\left(s_2\right) + \bar{\varepsilon}/4 \text{ and } x_t \in X. \tag{56}$$

Equation (55) states that all (informed and uninformed) agents with marginal rate of substitution above $(\kappa_t\left(s_1\right) - \bar{\varepsilon}/4)$ will receive a utility gain Δ from the trade z, in terms of current utility. Equation (56) states that all (informed and uninformed) agents with marginal rate of substitution below $(\kappa_t\left(s_1\right) + \bar{\varepsilon}/4)$ will receive a utility gain Δ from the trade $-z$, in terms of current utility.

Combining conditions (54) and (55) shows that in state s_1 there is at least $\alpha/2$ informed agents with after-trade utility above the long-run utility, $U(x_t + z, \delta_t) > \hat{v}_t$. Since all these agents would accept the trade z, this implies that the probability of acceptance of the trade is $\chi_t(z|s_1) > \alpha/2$.

Next, we want to show that the trade z is accepted with sufficiently low probability conditional on s_2. In particular, we want to show that $\chi_t(z|s_2) < \alpha/4$. The key step here is to make sure that the trade is rejected not only by informed but also by uninformed agents. The argument is that if this trade were to bo accepted by uninformed agents, then informed agents should be offering z and gaining in utility. Formally, proceeding by contradiction, suppose that the probability of z being accepted in state s_2 is large: $\chi_t(z|s_2) \geq \alpha/4$. Condition (54) implies that there is a positive mass of informed agents with $\mathcal{M}(x_t, \delta_t) < \kappa_t(s_2) + \bar{\varepsilon}/2$, $x_t \in X$, and close enough to the long-run utility $u_t \geq \hat{v}_t - \alpha\Delta/8$. By(56), these agents would be strictly better off making the offer z and consuming $x_t - z$ if the offer is accepted and consuming x_t if it is rejected, since

$$(1 - \chi_t(z_t|s_2))\, U(x_t, \delta_t) + \chi_t(z_t|s_2)\, U(x_t - z, \delta_t) > u_t + \alpha\Delta/4 > \hat{v}_t.$$

Since this strategy dominates the equilibrium payoff, this is a contradiction, proving that $\chi_t(z|s_2) < \alpha/4$.

Step 2. Consider the second case, in which the long run marginal rates of substitution of the informed agents are close to each other and there is a large enough mass of uninformed agents with marginal rate of substitution far from that of the informed agents.

The argument is as follows: with positive probability we can reach a point where it is possible to separate the marginal rates of substitution of a group of uninformed agents from the marginal rates of substitution of a group of informed agents. This means that the uninformed agents in the first group can make an offer z to the informed agents in the second group and they will accept the offer in *both* states s_1 and s_2. If the probabilities of acceptance $\chi_t(z|s_1)$ and $\chi_t(z|s_2)$ are sufficiently close to each other, this would be a profitable deviation for the uninformed, since their ex post beliefs after the offer is accepted would be close to their ex ante beliefs. In other words, in contrast to the previous case they would gain utility but not learn from the trade. It follows that the probabilities $\chi_t(z|s_1)$ and $\chi_t(z|s_2)$ must be sufficiently different in the two states, which leads to either (7) or to (8).

To formalize this argument, consider the expected utility of an uninformed agent with portfolio x_t and belief δ_t, who offers a trade z and stops trading from then on:

$$u_t + \delta_t \chi_t(z|s_1)\left(U(x_t - z, 1) - U(x_t, 1)\right) + (1 - \delta_t)\,\chi_t(z|s_2)\left(U(x_t - z, 0) - U(x_t, 0)\right),$$

where u_t is the expected utility if the offer is rejected and the following two terms are the expected gains if the offer is accepted, respectively, in states s_1 and s_2. This expected utility can be rewritten as

$$u_t+\chi_t\left(z|s_1\right)\left(U\left(x_t-z,\delta_t\right)-U\left(x_t,\delta_t\right)\right)+\left(1-\delta_t\right)\left(\chi_t\left(z|s_2\right)-\chi_t\left(z|s_1\right)\right)\left(U\left(x_t-z,0\right)-U\left(x_t,0\right)\right), \tag{57}$$

using the fact that $U\left(x_t,\delta_t\right)=\delta_t U\left(x_t,1\right)+\left(1-\delta_t\right)U\left(x_t,0\right)$ (by the definition of U). To interpret (57) notice that, if the probability of acceptance was independent of the signal, $\chi_t\left(z|s_1\right)=\chi_t\left(z|s_2\right)$, then the expected gain from making offer z would be equal to the second term: $\chi_t\left(z|s_1\right)\left(U\left(x_t-z,\delta_t\right)-U\left(x_t,\delta_t\right)\right)$. The third term takes into account that the probability of acceptance may be different in two states, i.e., $\chi_t\left(z|s_2\right)-\chi_t\left(z|s_1\right)$ may be different from zero. An alternative way of rearranging the same expression yields:

$$u_t+\chi_t\left(z|s_2\right)\left(U\left(x_t-z,\delta_t\right)-U\left(x_t,\delta_t\right)\right)+\left(1-\delta_t\right)\left(\chi_t\left(z|s_1\right)-\chi_t\left(z|s_2\right)\right)\left(U\left(x_t-z,1\right)-U\left(x,1\right)\right). \tag{58}$$

In the rest of the argument, we will use both (57) and (58).

Suppose that there exists a trade z and a period t which satisfy the following properties: (a) the probability that z is accepted in state 1 is large enough,

$$\chi_t\left(z|s_1\right)>\alpha/4,$$

and (b) there is a positive mass of uninformed agents with portfolios and beliefs that satisfy

$$u_t\geq\hat{v}_t-\left(\alpha/4\right)\Delta, \tag{59}$$

$$U\left(x_t-z,\delta_t\right)-U\left(x_t,\delta_t\right)\geq\Delta, \tag{60}$$

$$U\left(x_t-z,\delta\right)-U\left(x_t,\delta\right)\geq-\lambda\Delta \text{ for all } \delta\in\left[0,1\right], \tag{61}$$

for some $\Delta>0$ and $\lambda>0$. In words, the uninformed agents are sufficiently close to their long-run utility, their gains from trade at *fixed* beliefs have a positive lower bound Δ, and their gains from trade at *arbitrary* beliefs have a lower bound $-\lambda\Delta$.

Now we distinguish two cases. Suppose first that $\chi_t\left(z|s_2\right)\geq\chi_t\left(z|s_1\right)$. Then, for the uninformed agents who satisfy (59)-(61) the expected utility (57) is greater or equal than

$$\hat{v}_t-\left(\alpha/4\right)\Delta+\chi_t\left(z|s_1\right)\Delta-\left(\chi_t\left(z|s_2\right)-\chi_t\left(z|s_1\right)\right)\lambda\Delta.$$

From individual optimality, this expression cannot be larger than $\hat{v}_t$, since $\hat{v}_t$ is the maximum expected utility for a proposer in period t. We then obtain the following restriction on the acceptance probabilities $\chi_t\left(z|s_1\right)$ and $\chi_t\left(z|s_2\right)$:

$$\chi_t\left(z|s_1\right)\left(1+\lambda\right)\Delta\leq\alpha\Delta/4+\chi_t\left(z|s_2\right)\lambda\Delta.$$

Since $\chi_t(z|s_1) > \alpha/2$ and $\chi_t(z|s_1) \geq \chi_t(z|s_2)$ it follows that $\alpha/4 < (1/2)\chi_t(z|s_2)$ and we obtain

$$\chi_t(z|s_1)(1+\lambda) \leq \chi_t(z|s_2)(1/2+\lambda),$$

which is equivalent to

$$\chi_t(z|s_1) \geq \frac{1+\lambda}{1/2+\lambda}\chi_t(z|s_2). \tag{62}$$

This shows that the probability of acceptance in state s_1 is larger than the probability of acceptance in state s_2 by a factor $(1+\lambda)/(1/2+\lambda)$ greater than 1.

Consider next the case $\chi_t(z|s_2) < \chi_t(z|s_1)$. Then, for the uninformed agents who satisfy (59)-(61) the expected utility (58) is greater or equal than

$$\hat{v}_t - \alpha\Delta/4 + \chi_t(z|s_2)\Delta - (\chi_t(z|s_1) - \chi_t(z|s_2))\lambda\Delta.$$

An argument similar to the one above shows that optimality requires

$$\chi_t(z|s_2) \geq \frac{1+\lambda}{1/2+\lambda}\chi_t(z|s_1).$$

Some algebra shows that this inequality and $\chi_t(z|s_1) > \alpha/2$ imply

$$1 - \chi_t(z|s_1) > 1 - \frac{\alpha}{2}\frac{1/2+\lambda}{1+\lambda}, \tag{63}$$

$$1 - \chi_t(z|s_1) > \frac{(1-\alpha/2)(1/2+\lambda)}{(1-\alpha/2)(1/2+\lambda) - \alpha/4}(1 - \chi_t(z|s_2)), \tag{64}$$

giving us a positive lower bound for the probability of rejection $1-\chi_t(z|s_1)$ and showing that $1-\chi_t(z|s_1)$ exceeds $1-\chi_t(z|s_2)$ by a factor greater than 1.

To complete this step, we show that there exists a trade z and a period t which satisfy properties (a) and (b).

Notice that $P(|\mathcal{M}(x_t,\delta_t) - \kappa_t(s_1)| \geq 2\bar{\varepsilon} \mid s_1) \geq \bar{\varepsilon}$ requires that either $P(\mathcal{M}(x_t,\delta_t) \leq \kappa_t(s_1) - 2\bar{\varepsilon} \mid s_1) \geq \bar{\varepsilon}/2$ holds or $P(\mathcal{M}(x_t,\delta_t) \geq \kappa_t(s_1) + 2\bar{\varepsilon} \mid s_1) \geq \bar{\varepsilon}/2$. We concentrate on the first case, as the second is treated symmetrically. Set the trading price at $p = \min\{\kappa(s_1), \kappa(s_2)\} - \bar{\varepsilon}/2$. Lemma 5 implies that there are positive scalars Δ and λ and a trade $z = (\zeta, -p\zeta)$ with $\|z\| < \theta$ that satisfies the following inequalities:

$$U(x_t - z, \delta_t) \geq u_t + \Delta,\ U(x_t - z, \delta) \geq u_t - \lambda\Delta \text{ for all } \delta \in [0,1], \quad \text{if } \mathcal{M}(x_t,\delta_t) < p - \bar{\varepsilon}/4 \text{ and } x_t \in X, \tag{65}$$

and

$$U(x_t + z, \delta_t) \geq u_t + \Delta \text{ if } \mathcal{M}(x_t,\delta_t) > p + \bar{\varepsilon}/4 \text{ and } x_t \in X. \tag{66}$$

Since $|\mathcal{M}(x_t, \delta_t) - \kappa_t(s_1)| < \bar{\varepsilon}/4$ implies $\mathcal{M}(x_t, \delta_t) > \kappa_t(s_1) - \bar{\varepsilon}/4$ and $\kappa_t(s_1) - \bar{\varepsilon}/4$ is larger than $p + \bar{\varepsilon}/4$ by construction, conditions (54) and (66) guarantee that there is a positive mass of informed agents who accept z, ensuring that $\chi_t(z|s_1) > \alpha/2$, showing that z satisfies property (a).

Next, we want to prove that there is a positive mass of uninformed agents who gain from making offer z. To do so, notice that $|\kappa_t(s_1) - \kappa_t(s_2)| < \bar{\varepsilon}$ implies

$$p - \bar{\varepsilon}/4 = \min\{\kappa_t(s_1), \kappa_t(s_2)\} - (3/4)\bar{\varepsilon} \geq \kappa_t(s_1) - (7/4)\bar{\varepsilon} > \kappa_t(s_1) - 2\bar{\varepsilon},$$

which implies

$$P(\mathcal{M}(x_t, \delta_t) < p - \bar{\varepsilon}/4 \mid s_1) \geq P(\mathcal{M}(x_t, \delta_t) \leq \kappa_t(s_1) - 2\bar{\varepsilon} \mid s_1) \geq \bar{\varepsilon}/2.$$

This, using Lemma 3 and condition (53), implies

$$P(\mathcal{M}(x_t, \delta_t) < p - \bar{\varepsilon}/4, u_t \geq \hat{v}_t - \alpha\Delta/4, x_t \in X \mid s_1) > 0,$$

which, combined with (65), shows that the trade z satisfies property (b).

Step 3. Here we put together the bounds established above and define the scalars β and ρ in the lemma's statement. Consider the case treated in Step 1. In this case, we can find a trade z such that the probability of acceptance conditional on each signal satisfies: $\chi_t(z|s_1) > \alpha/2$ and $\chi_t(z|s_2) < \alpha/4$. Therefore, in this case condition (7) is true as long as β and ρ satisfy

$$\beta \leq \alpha/2 \text{ and } \rho \leq 2.$$

Consider the case treated in Step 2. In this case, we can find a trade z such that either $\chi_t(z|s_1) > \alpha/2$ and (62) hold or (63) and (64) hold. This implies that either condition (7) or condition (8) hold, as long as β and ρ satisfy

$$\beta \leq 1 - \frac{\alpha}{2}\frac{1/2 + \lambda}{1 + \lambda}, \rho \leq \frac{1 + \lambda}{1/2 + \lambda}, \rho \leq \frac{(1 - \alpha/2)(1/2 + \lambda)}{(1 - \alpha/2)(1/2 + \lambda) - \alpha/4}.$$

Setting

$$\begin{aligned}
\beta &= \min\left\{\alpha/2, 1 - \frac{\alpha}{2}\frac{1/2 + \lambda}{1 + \lambda}\right\} > 0, \\
\rho &= \min\left\{2, \frac{1 + \lambda}{1/2 + \lambda}, \frac{(1 - \alpha/2)(1/2 + \lambda)}{(1 - \alpha/2)(1/2 + \lambda) - \alpha/4}\right\} > 1,
\end{aligned}$$

ensures that all the conditions above are satisfied, completing the proof. ■

7.5 Proof of Proposition 4

Proposition 4 states that marginal rates of substitution converge for uninformed agents. The proof is by contradiction and relies on constructing a deviation that yields a positive utility gain for the uninformed agents if marginal rates of substitution fail to converge. This deviation consists of making a sequence of offers in periods T to $T+J$. The first $T+J-1$ offers, denoted by the sequence $\{\hat{z}_j\}_{j=0}^{J-1}$, allow the agent to learn the signal with arbitrary precision. This is the experimentation stage. If the agent receives the appropriate sequence of responses to these $J-1$ offers, the agent makes one final offer, denoted z^*, which gives him a positive utility gain by trading with the informed agents.

Two preliminary results need to be established first: Lemma 7 and Lemma 8. Lemma 7 shows that the beliefs of uninformed agents δ_t stay away from zero when the signal is s_1. That is, the uninformed agents can only be very wrong with a small probability. This result is used to ensure that when the uninformed agent deviates and enters the experimentation phase to learn signal s_1, his ex post beliefs will converge to 1 with positive probability.

Lemma 7 *For all $\varepsilon > 0$ the probability that the belief δ_t is above the threshold $\varepsilon/(1+\varepsilon)$ conditional on signal s_1 is bounded below for all t:*

$$P\left(\delta_t \geq \varepsilon/\left(1+\varepsilon\right) \mid s_1\right) > 1-\varepsilon.$$

Proof. Since $\delta_t(\omega)$ are equilibrium beliefs, Bayesian rationality requires $P(s_1 \mid \delta_t < \varepsilon/(1+\varepsilon)) < \varepsilon/(1+\varepsilon)$ for all $\varepsilon > 0$. The latter condition implies $P(s_2 \mid \delta_t < \varepsilon/(1+\varepsilon)) > 1-\varepsilon/(1+\varepsilon)$ and thus

$$\frac{P\left(s_1 \mid \delta_t < \varepsilon/\left(1+\varepsilon\right)\right)}{P\left(s_2 \mid \delta_t < \varepsilon/\left(1+\varepsilon\right)\right)} < \varepsilon,$$

for all $\varepsilon > 0$. Bayes' rule implies that

$$\frac{P\left(s_1 \mid \delta_t < \varepsilon/\left(1+\varepsilon\right)\right)}{P\left(s_2 \mid \delta_t < \varepsilon/\left(1+\varepsilon\right)\right)} = \frac{P\left(\delta_t < \varepsilon/\left(1+\varepsilon\right) \mid s_1\right)P\left(s_1\right)}{P\left(\delta_t < \varepsilon/\left(1+\varepsilon\right) \mid s_2\right)P\left(s_2\right)}.$$

Combining the last two equations and using $P(s_1) = P(s_2) = 1/2$ yields

$$P\left(\delta_t < \varepsilon/\left(1+\varepsilon\right) \mid s_1\right) < \varepsilon P\left(\delta_t < \varepsilon/\left(1+\varepsilon\right) \mid s_2\right) \leq \varepsilon,$$

which gives the desired inequality.

Lemma 8 is a stronger version of the convergence result for the marginal rates of substitution of informed agents (Proposition 1). It shows that the series $\kappa_t(s)$ (the long-run marginal rate of substitution of informed agents) is approximately constant over fixed intervals of length J, for any choice of the length J. This implies that the marginal rates of substitution of informed agents at time $t+J$ are close to the value $\kappa_t(s)$, if we choose t large enough. This property will be useful when constructing the final offer z^* made by the deviating uninformed agent in the proof of Proposition 4. ■

Lemma 8 *For any integer J, the sequence $\kappa_t(s_1)$ satisfies the property:*

$$\lim_{t\to\infty} |\kappa_{t+J}(s_1) - \kappa_t(s_1)| = 0.$$

For all $\varepsilon > 0$ and all integers J it is possible to find a T such that

$$P(|\mathcal{M}(x_{t+J}, \delta_{t+J}) - \kappa_t(s_1)| < \varepsilon, \delta_{t+J} = 1 \mid s) > \alpha - \varepsilon \text{ for all } t \geq T.$$

Proof. Let us begin from the first part of the lemma. Suppose, by contradiction, that

$$|\kappa_{t+J}(s_1) - \kappa_t(s_1)| > \varepsilon$$

for some $\varepsilon > 0$ for infinitely many periods. Then, at some date t, an informed agent with marginal rate of substitution close to $\kappa_t(s)$ can find a profitable deviation by holding on to his portfolio x_t for J periods and then trade with other informed agents at $t+J$. Let us formalize this argument. Suppose, without loss of generality, that

$$\kappa_{t+J}(s_1) > \kappa_t(s_1) + \varepsilon$$

for infinitely many periods (the other case is treated in a symmetric way). Next, using our usual steps and Proposition 1, it is possible to find a compact set X, a time T, and a utility gain $\Delta > 0$ such that the following two properties are satisfied: (i) in all periods $t \geq T$ there is at least a measure $\alpha/2$ of informed agents with marginal rate of substitution sufficiently close to $\kappa_t(s)$, utility close to its long run level, and portfolio x_t in X, that is,

$$P\left(|\mathcal{M}(x_t, \delta_t) - \kappa_t(s)| < \varepsilon/3, u_t \geq \hat{v}_t - \gamma^J \alpha\Delta/2, x_t \in X, \delta_t = 1 \mid s\right) > \alpha/2, \tag{67}$$

and (ii) in all periods $t \geq T$ in which $\kappa_{t+J}(s) > \kappa_t(s) + \varepsilon$ there is a trade z such that

$$U(x - z, 1) > U(x, 1) + \Delta \text{ if } \mathcal{M}(x, 1) < \kappa_t(s) + \varepsilon/3 \text{ and } x \in X, \tag{68}$$

and

$$U(x + z, 1) > U(x, 1) + \Delta \text{ if } \mathcal{M}(x, 1) > \kappa_{t+J}(s) - \varepsilon/3 \text{ and } x \in X. \tag{69}$$

Pick a time $t \geq T$ in which $\kappa_{t+J}(s) > \kappa_t(s) + \varepsilon$ and consider the following deviation. Whenever an informed agent reaches time t and his portfolio x_t satisfies $\mathcal{M}(x_t, 1) < \kappa_t(s) + \varepsilon/3$ and $x_t \in X$, he stops trading for J periods and then makes an offer z that satisfies (68) and (69). If the offer is rejected he stops trading from then on. The probability that this offer is accepted at time $t+J$ must satisfy $\chi_{t+J}(z|s_1) > \alpha/2$, because of conditions (67) and (69). Therefore, the expected utility from this strategy, from the point of view of time t is

$$u_t + \gamma^J \chi_{t+J}(z|s_1)(U(x_t - z, 1) - u_t) > u_t + \gamma^J \alpha\Delta/2 \geq \hat{v}_t,$$

so this strategy is a profitable deviation and we have a contradiction.

The second part of the lemma follows from the first part, using Proposition 1 and the triangle inequality. ■

Proof of Proposition 4. Suppose, by contradiction, that there exist an $\varepsilon > 0$ such that for some state $s \in \{s_1, s_2\}$ the following condition holds for infinitely many t:

$$P\left(\left|\mathcal{M}\left(x_t, \delta^I(s)\right) - \kappa_t(s)\right| > \varepsilon \mid s\right) > \varepsilon,$$

where $\mathcal{M}\left(x_t, \delta^I(s)\right)$ is the marginal rate of substitution of an agent (informed or uninformed) evaluated at the belief of the informed agents $\delta^I(s)$. In other words, it is the marginal rate of substitution evaluated as if the agent knew the true signal.

Without loss of generality, let us focus on state s_1 and suppose

$$P\left(\mathcal{M}(x_t, 1) - \kappa_t(s_1) > \varepsilon \mid s_1\right) > \varepsilon \tag{70}$$

for infinitely many t. The other case is treated in a symmetric way.

We want to show that if (70) holds, we can find a profitable deviation for the uninformed agent. In particular, we consider a deviation of this form:

(i) The player follows the equilibrium strategy $\boldsymbol{\sigma}$ up to period T.

(ii) At time T, if his portfolio satisfies $\mathcal{M}(x_T, 1) > \kappa_t(s) + \varepsilon$ and his beliefs δ_T is above some positive lower bound $\underline{\delta}$ (and some other technical conditions are satisfied) he goes on to the experimentation stage (iii), otherwise, he keeps playing $\boldsymbol{\sigma}$.

(iii) The experimentation stage lasts between T and $T+J-1$. An agent makes the sequence of offers $\{\hat{z}_j\}_{j=0}^{J-1}$ as long as he is selected as the proposer. The "favorable" responses to the offers $\{\hat{z}_j\}_{j=0}^{J-1}$ are given by the binary sequence $\{\hat{r}_j\}_{j=0}^{J-1}$. If at any point during the experimentation stage the agent is not selected as the proposer or fails to receive response $\hat{r}_j$ after offer $\hat{z}_j$, he stops trading. Otherwise, he goes to (iv).

(iv) At time $T+J$, after making all the offers $\{\hat{z}_j\}_{j=0}^{J-1}$ and receiving responses equal to $\{\hat{r}_j\}_{j=0}^{J-1}$, if the player is selected as the proposer one more time he offers z^* and stops trading at $T+J+1$. Otherwise, he stops trading right away.

The expected payoff of this strategy, from the point of view of a deviating agent at time T, is

$$\begin{aligned} w &= u_T - \hat{L} + \delta_T \gamma^J 2^{-J-1} \xi_1 \chi_{T+J}\left(z^*|s_1\right) \left[U(x_T + \sum_{j=0}^{J-1} \hat{z}_j + z^*, 1) - U(x_T, 1) \right] + \\ &\quad + \left(1-\delta_T\right) \gamma^J 2^{-J-1} \xi_2 \chi_{T+J}\left(z^*|s_2\right) \left[U(x_T + \sum_{j=0}^{J-1} \hat{z}_j + z^*, 0) - U(x_T, 0) \right], \end{aligned} \tag{71}$$

where the term $\hat{L}$ captures the expected utility losses if the player makes some or all of the offers in $\{\hat{z}_j\}_{j=0}^{J-1}$ but not the last offer z^* and the following two terms capture the expected utility gains in states s_1 and s_2, if all the deviating offers, including z^*, are accepted. The factors ξ_1 and ξ_2 denote the probabilities in states s_1 and s_2, that player receives the sequence of responses $\{\hat{r}_j\}_{j=0}^{J-1}$. Notice that γ^J is the probability that the game does not end between periods T and $T+J$ and 2^{-J-1} is the probability of being selected as the proposer in all these periods.

In order to show that the strategy above is a profitable deviation, we need to show that the utility gain in the first square brackets is large enough, by choosing z^* to be a profitable trade with informed agents in s_1, and that the remaining terms are sufficiently small. In the rest of the proof, we choose the time T, the lower bound $\underline{\delta}$, and the offers $\{\hat{z}_j\}_{j=0}^{J-1}$ and z^* to achieve this goal.

Step 1. (Bounds on gains and losses for the final trade) Following steps similar to the ones in the proof of Proposition 1, we can use Lemmas 4 and 7 to find a compact set $X \subset R^2_{++}$ and a period T' such that

$$P\left(\delta_t \geq \underline{\delta}, x_t \in X \mid s_1\right) > 1 - \varepsilon/2 \tag{72}$$

for all $t \geq T'$, where $\underline{\delta} = (\varepsilon/2) / (1 + \varepsilon/2) > 0$. Pick a scalar $\theta^* > 0$ such that $x + z > 0$ when $x \in X$ and $\|z\| < \theta^*$. Using Lemma 5, we can then find a $\Delta^* > 0$ which is a lower bound for the gains from trade between two agents with marginal rates of substitution differing by at least $\varepsilon/2$ and portfolios in X, making trades of norm smaller than θ^*. This will be used as a lower bound for the gains from trading in state s_1. Define an upper bound for the potential losses of an uninformed agent who makes a trade of norm smaller than or equal to θ^* in the other state, s_2:

$$L^* \equiv - \min_{x \in X, \|z\| \leq \theta^*} \left\{ U\left(x+z,0\right) - U\left(x,0\right) \right\}.$$

Next, choose J to be an integer large enough that

$$\underline{\delta}\left(\alpha/2\right)\Delta^* - \left(1-\underline{\delta}\right)\rho^{-J} L^* > 0,$$

where ρ is the scalar defined in Proposition 3. This choice of J ensures that the experimentation phase is long enough that, when offering the last trade, the agent assigns sufficiently high probability to state s_1, so that the potential gain Δ^* dominates the potential loss L^*.

Step 2. *(Bound on losses from experimentation)* To simplify notation, let

$$\tilde{\Delta} = \gamma^J 2^{-J-1} \beta^J \left(\underline{\delta} \left(\alpha/2 \right) \Delta^* - \left(1 - \underline{\delta} \right) \rho^{-J} L^* \right),$$

where β is the positive scalar defined in Proposition 3. Choose a scalar $\hat{\theta} > 0$ such that for all $x \in X$, all $\|z_1\| < J\hat{\theta}$, all $\|z_2\| \leq \theta^*$, and any $\delta \in [0, 1]$ the following inequality holds

$$|U(x + z_1 + z_2, \delta) - U(x + z_2, \delta)| < \tilde{\Delta}/3. \tag{73}$$

Next, applying Proposition 3 we can find a time $T'' \geq T'$ such that in all $t \geq T''$ there is a trade of norm smaller than $\hat{\theta}$ that satisfies either (7) or (8). Before using this property to define the offers $\{\hat{z}_j\}_{j=0}^{J-1}$, we need to define the time period T where the deviation occurs. To do so, using our starting hypothesis (70), condition (72), and applying Lemma 2, we can find a $T''' \geq T''$ such that for infinitely many periods $t \geq T'''$ there is a positive mass of uninformed agents who have: marginal rate of substitution sufficiently above $\kappa_t(s_1)$, utility near its long-run level, beliefs sufficiently favorable to s_1, and portfolio in X; that is,

$$P\left(\mathcal{M}(x_t, 1) - \kappa_t(s_1) > \varepsilon, \delta_t \geq \underline{\delta}, u_t > \hat{v}_t - \tilde{\Delta}/3, x_t \in X \mid s_1\right) > 0. \tag{74}$$

Finally, applying Lemma 8, we pick a $T \geq T'''$ so that (74) holds at $t = T$ *and*, at time $T + J$, there is a sufficiently large mass of informed agents who have: marginal rate of substitution sufficiently near $\kappa_T(s_1)$, utility near its long-run level, and portfolio in X; that is,

$$P\left(|\mathcal{M}(x_{T+J}, 1) - \kappa_T(s_1)| < \varepsilon/2, \delta_{T+J} = 1, u_{T+J} > \hat{v}_{T+J} - \Delta/2, x_{T+J} \in X \mid s_1\right) > \alpha/2. \tag{75}$$

Having defined T, we can apply Proposition 3 to find the desired sequence of trades $\{\hat{z}_j\}_{j=0}^{J-1}$ of norm smaller than $\hat{\theta}$, that satisfy either (7) or (8). For each trade $\hat{z}_j$, if (7) holds we set $\hat{r}_j = 1$ (accept). In this way the probability of observing $\hat{r}_j$ is $\chi_{T+j}(\hat{z}_j|s_1) > \beta$ in state s_1 and $\chi_{T+j}(\hat{z}_j|s_2) < \rho^{-1}\chi_{T+j}(\hat{z}_j|s_1)$ in state s_2. Otherwise, if (8) holds, we set $\hat{r}_j = 0$ and obtain analogous inequalities. This implies that the factors ξ_1 and ξ_2 in (71) satisfy

$$\xi_1 > \beta^J \text{ and } \xi_2 < \xi_1 \rho^{-J}. \tag{76}$$

Step 3. (Define z^* and check profitable deviation) We can now define the final trade z^* to be a trade of norm smaller than θ^*, such that

$$\begin{aligned} U(x - z^*, 1) &> U(x, 1) + \Delta^* \text{ if } \mathcal{M}(x, 1) > \kappa_T(s_1) + \varepsilon \text{ and } x \in X, \\ U(x + z^*, 1) &> U(x, 1) + \Delta^* \text{ if } \mathcal{M}(x, 1) < \kappa_T(s_1) + \varepsilon/2 \text{ and } x \in X, \end{aligned}$$

which is possible given the definition of Δ^*. Finally, we check that we have constructed a profitable deviation. Let uninformed agents start deviating whenever the following conditions are satisfies at date T:

$$\mathcal{M}(x_T, 1) > \kappa_T(s_1) + \varepsilon, \delta_T \geq \underline{\delta}, u_T > \hat{v}_T - \tilde{\Delta}/3, x_T \in X.$$

Equation (74) shows that this happens with positive probability. Let us evaluate the deviating strategy payoff (71), beginning with the last two terms. The triangle inequality implies $\left\|\sum_{j=0}^{J-1} \hat{z}_j\right\| < J\hat{\theta}$. Then the definition of z^* and (73) imply that the gain from trade of the uninformed agent, conditional on s_1, is bounded below:

$$\begin{aligned} U(x_T + \sum_{j=0}^{J-1} \hat{z}_j + z^*, 1) - U(x_T, 1) &\geq U(x_T + z^*, 1) - U(x_T, 1) - \left| U(x_T + \sum_{j=0}^{J-1} \hat{z}_j + z^*, 1) - U(x_T + z^*, 1) \right| \\ &> \Delta^* - \tilde{\Delta}/3. \end{aligned}$$

The definition of L^* implies that the gain conditional on s_2 is also bounded:

$$U(x_T + \sum_{j=0}^{J-1} \hat{z}_j + z^*, 0) - U(x_T, 0) > -L^* - \tilde{\Delta}/3.$$

Moreover, condition (75) shows that the probability that informed agents accept z^* at $T + J$ satisfies $\chi_{T+J}(z^*|s_1) > \alpha/2$. These results, together with the inequalities (76) and the fact that $\chi_{T+J}(z^*|s_2) \leq 1$, imply that the last two terms in (71) are bounded below by

$$\gamma^J 2^{-J-1} \beta^J \left[\underline{\delta} (\alpha/2) \left(\Delta^* - \tilde{\Delta}/3 \right) - (1 - \underline{\delta}) \rho^{-J} \left(L^* + \tilde{\Delta}/3 \right) \right],$$

which, by the definition of $\tilde{\Delta}$, is greater than $(2/3)\tilde{\Delta}$. Finally, all the expected losses in $\hat{L}$ in (71) are bounded above by $\tilde{\Delta}/3$, thanks to (73). Therefore, $w > u_T + \tilde{\Delta}/3$. Since $u_T > \hat{v}_T - \tilde{\Delta}/3$, we conclude that $w > \hat{v}_T$ and we have found a profitable deviation. ■

7.6 Proof of Theorem 1

We begin from the second part of the theorem, proving (11), which characterizes the limit behavior of $\kappa_t(s)$.

Without loss of generality, let $s = s_1$. Suppose first that for infinitely many periods the long run marginal rate of substitution $\kappa_t(s_1)$ is larger than the ratio of the probabilities $\phi(s_1)/(1 - \phi(s_1))$ by a factor larger than $1 + \varepsilon$:

$$\kappa_t(s_1) > (1 + \varepsilon) \phi(s_1)/(1 - \phi(s_1)) \text{ for some } \varepsilon > 0.$$

Proposition 4 then implies that for all $\eta > 0$ and T there is a t such that almost all agents have portfolios that satisfy $u'(x_t^1)/u'(x_t^2) \geq 1 + \varepsilon/2$:

$$P\left(u'(x_t^1)/u'(x_t^2) \geq 1 + \varepsilon/2 \mid s_1\right) > 1 - \eta. \tag{77}$$

We want to show that this property violates uniform market clearing, since it implies that almost all agents hold more of asset 2 than of asset 1.

Uniform market clearing implies that for any $\zeta > 0$ we can find an M such that

$$\int_{x_t^1(\omega)\leq m} x_t^1(\omega)\, dP(\omega|s_1) \geq 1 - \zeta \text{ for all } m \geq M \text{ and all } t. \tag{78}$$

Moreover, since $\int x_t^2(\omega)\, dP(\omega|s_1) = 1$, this implies that

$$\int_{x_t^1(\omega)\leq m} \left(x_t^2(\omega) - x^1(\omega)\right) dP(\omega|s_1) \leq \zeta \text{ for all } m \geq M \text{ and all } t. \tag{79}$$

The idea of the proof is to reach a contradiction by splitting the integral on the left-hand side of (79) in three pieces: a group of agents with a strictly positive difference $x_t^2 - x_t^1$, a group of agents with a non-negative difference $x_t^2 - x_t^1$, and a small residual group. The argument here follows a similar logic as the proof of Proposition 2.

Using Lemma 4, find a compact set X and a period T such that for all $t \geq T$ at least half of the agents have portfolios in X:

$$P(x_t \in X \mid s_1) \geq 1/2 \text{ for all } t \geq T. \tag{80}$$

Let us then find a lower bound for the difference between the holdings of asset 1 and 2 for agents with portfolios in X that satisfy $u'(x_t^1)/u'(x_t^2) \geq 1 + \varepsilon/2$. We do so by solving the problem

$$\begin{aligned} d &= \min_{x\in X} \left(x^2 - x^1\right) \\ &\quad \text{s.t. } u'(x^1)/u'(x^2) \geq 1 + \varepsilon/2, \end{aligned}$$

which gives a $d > 0$.

Let us pick $\zeta = d/5$ and find an M such that (78) and (79) hold. Condition (79) (with $\zeta = d/5$) is the market clearing condition that we will contradict below. Condition (78) is also useful, because it gives us a lower bound for $P\left(x_t^1 \leq m\right)$:

$$P\left(x_t^1 \leq m\right) \geq 1 - \zeta/m \text{ for all } m \geq M \text{ and all } t, \tag{81}$$

which follows from the chain of inequalities

$$mP\left(x_t^1 > m\right) \leq \int_{x_t^1(\omega)>m} x_t^1(\omega)\, dP(\omega|s_1) \leq \zeta.$$

Using our hypothesis (77) we know that for any $\eta > 0$ we can find a period $t \geq T$ in which more than $1-\eta$ agents satisfy $u'(x_t^1)/u'(x_t^2) \geq 1+\varepsilon/2$. Combining this with (80) and (81) (applying Lemma 3), we can always find a $t \geq T$ in which almost all agents satisfy $u'(x_t^1)/u'(x_t^2) \geq 1+\varepsilon/2$ and $x_t \leq m$:

$$P\left(u'(x_t^1)/u'(x_t^2) \geq 1+\varepsilon/2, x_t \leq m \mid s_1\right) > 1-\eta-\zeta/m, \tag{82}$$

and almost half of them satisfy $u'(x_t^1)/u'(x_t^2) \geq 1+\varepsilon/2$ and $x_t \leq m$, and have portfolios in X:

$$P\left(u'(x_t^1)/u'(x_t^2) \geq 1+\varepsilon/2, x_t \leq m, x_t \in X \mid s_1\right) > 1/2-\eta-\zeta/m. \tag{83}$$

Define the three disjoint sets

$$\begin{aligned}
A_1 &= \left\{\omega : u'(x_t^1)/u'(x_t^2) \geq 1+\varepsilon/2, x_t \in X, x_t^1 \leq m\right\}, \\
A_2 &= \left\{\omega : u'(x_t^1)/u'(x_t^2) \geq 1+\varepsilon/2, x_t^1 \leq m\right\}/A_1, \\
A_3 &= \left\{\omega : u'(x_t^1)/u'(x_t^2) < 1+\varepsilon/2, x_t^1 \leq m\right\},
\end{aligned}$$

which satisfy $A_1 \cup A_2 \cup A_3 = \left\{\omega : x_t^1 \leq m\right\}$. We can then bound from below the following three integrals:

$$\begin{aligned}
\int_{A_1} \left(x_t^2 - x_t^1\right) dP(\omega|s_1) &\geq d \cdot (1/2-\eta-\zeta/m), \\
\int_{A_2} \left(x_t^2 - x_t^1\right) dP(\omega|s_1) &\geq 0, \\
\int_{A_3} \left(x_t^2 - x_t^1\right) dP(\omega|s_1) &\geq -m \cdot (\eta+\zeta/m).
\end{aligned}$$

The first inequality follows from the definitions of d and A_1 and the fact that $P(A_1|s_1) > 1/2-\eta-\zeta/m$ from (83). The second follows from the definition of A_2 and the fact that $u'(x_t^1)/u'(x_t^2) > 1$ implies $x_t^2 > x_t^1$. The third follows from the definition of A_3 (which implies $x_t^2 - x_t^1 \geq -m$) and the fact that $P(A_3|s_1) < \eta + \zeta/m$ from (82). Summing term by term, we then obtain

$$\int_{x_t^1(\omega)\leq m} \left(x_t^2 - x_t^1\right) dP(\omega|s_1) \geq d \cdot (1/2-\eta-\zeta/m) - m \cdot (\eta+\zeta/m).$$

Since we can choose an m arbitrarily large and an η arbitrarily close to 0 (in that order), we can make this expression as close as we want to $d/2-\zeta$ which is strictly greater than ζ, given that $\zeta = d/5 < d/4$. This contradicts the market clearing condition (79).

In a similar way we can rule out the case in which $\kappa_t(s_1) < (1-\varepsilon)\,\phi(s_1)/(1-\phi(s_1))$ for infinitely many periods. This completes the argument for $\lim_{t\to\infty}\kappa_t(s_1) = \phi(s_1)/(1-\phi(s_1))$. An analogous argument can be applied to s_2.

To complete the proof, we need to prove the long run efficiency of equilibrium portfolios, i.e., property (10). Proposition 4 and $\lim \kappa_t(s) = \phi(s)/(1-\phi(s))$, imply, by the properties of convergence in probability, that

$$\lim_{t\to\infty} P\left(\left|u'\left(x_t^1\right)/u'\left(x_t^2\right)-1\right| > \varepsilon\right) = 0. \tag{84}$$

We want to show that negating (10) leads to a contradiction of (84).

Suppose that for some $\varepsilon > 0$

$$P\left(\left|x_t^1 - x_t^2\right| > \varepsilon\right) > \varepsilon$$

holds for infinitely many periods. Then, as usual, we can use Lemmas 3 and 4 to find a compact set X such that the following condition holds for infinitely many periods:

$$P\left(\left|x_t^1 - x_t^2\right| > \varepsilon, x_t \in X\right) > \varepsilon/2.$$

But then the continuity of $u'(\cdot)$ implies that there is a $\delta > 0$ such that

$$\left|u'\left(x^1\right)/u'\left(x^2\right)-1\right| > \delta \implies \left|x^1 - x^2\right| > \varepsilon \text{ for all } x \in X$$

which implies

$$P\left(\left|u'\left(x_t^1\right)/u'\left(x_t^2\right)-1\right| > \delta, x_t \in X\right) \geq P\left(\left|x_t^1 - x_t^2\right| > \varepsilon, x_t \in X\right) > \varepsilon/2.$$

Given that

$$P\left(\left|u'\left(x_t^1\right)/u'\left(x_t^2\right)-1\right| > \delta\right) \geq P\left(\left|u'\left(x_t^1\right)/u'\left(x_t^2\right)-1\right| > \delta, x_t \in X\right)$$

we conclude that there are $\varepsilon, \delta > 0$ such that

$$P\left(\left|u'\left(x_t^1\right)/u'\left(x_t^2\right)-1\right| > \delta\right) > \varepsilon/2,$$

contradicting (84) and completing the proof.

7.7 Proof of Theorem 3

Proof. Proceeding by contradiction, assume that the value of information is zero. We then derive a contradiction in two steps. First, we show that in some period t there is an offer $z \neq 0$ which is optimal for a proposer with endowment $x_{0,i}$, independently of whether he is uninformed or informed. Second, we construct a set of offers that are strictly preferred to z by all responders, and use this set to argue that the offer z cannot be optimal for the informed both after observing signal s_1 and after observing signal s_2.

Step 1. Under our hypothesis of zero value of information, we want to show that there exists an offer $z \neq 0$ which is made with positive probability by an uninformed proposer with endowment $x_{0,i}$ at time 0 and which is optimal for an informed proposer, independently of whether he receives signal s_1 or s_2.

First, notice that in some period t a positive measure of agents must make a non-zero offer which is accepted with positive probability and yields a strictly positive expected utility gain to either the proposer or the responder. Otherwise, all agents' expected utility at date zero would be equal to their expected utility under zero trade. This leads to a contradiction because, under the assumption that the initial allocation is inefficient ($x^1_{0,i} \neq x^2_{0,i}$) and given the long run efficiency result in Theorem 1, the expected utility at date 0 (before agents are assigned endowments and information) must be strictly greater than the expected utility under zero trade, that is, we must have $E[v_0] > E[U(x_0, 1/2)]$. Without loss of generality, suppose such a non-zero offer is made in period $t = 1$ (the argument can be easily translated to the first period when such an offer is made).

Suppose, for the moment, that this non-zero offer is made by an uninformed proposer with portfolio $x_{0,i}$. Since the value of information is zero, it must also be optimal for an informed proposer with the same portfolio to follow the same strategy, irrespective of whether he observes s_1 or s_2. This follows from a standard result, which we prove here for completeness. Let $W(\sigma|s)$ denote the expected utility at time 1 of an agent with endowment $x_{0,i}$ who follows strategy σ conditional on the signal being s. If the value of information is zero it means that

$$\max_{\tilde{\sigma}} \sum_s (1/2)\, W(\tilde{\sigma}|s) = \sum_s (1/2) \max_{\tilde{\sigma}} W(\tilde{\sigma}|s). \tag{85}$$

Suppose σ^* solves the problem on the left-hand side. Then, by definition,

$$\max_{\tilde{\sigma}} W\left(\sigma'|s\right) \geq W(\sigma^*|s) \text{ for all } s.$$

However, if any of these holds with a strict inequality, (85) would be violated. It follows that $W(\sigma^*|s) = \max_{\tilde{\sigma}} W(\sigma'|s)$ for all s, showing that σ^* is optimal for both informed agents.

To complete this step, we need to show that whenever a non-zero offer is made by some agent, there must be a non-zero offer made by an uninformed agent. We divide the argument in two cases:

Case 1. Suppose it is part of the equilibrium strategy for an informed agent with initial portfolio x to make offer $z \neq 0$ at date 0 after signal $s = s_1$ and the offer gives the proposer a strictly positive utility gain, that is, $V_1(x - z, 1) > V_1(x, 1)$ (the case $s = s_2$ is analogous). We want to show that then there must be some other offer $z' \neq 0$ which is made with positive probability by the uninformed proposer at date 0.

Let $V_0^P(x, \delta)$ denote the expected utility of an agent with portfolio x and belief δ, contingent on the agent being a proposer at date 0. Then, the strict optimality of z implies that

$$\begin{aligned} V_0^P(x,1) &= (1-\gamma)U(x,1) + \gamma\left[\chi_0(z|s_1)V_1(x-z,1) + (1-\chi_0(z|s_1))V_1(x,1)\right] \\ &> (1-\gamma)U(x,1) + \gamma V_1(x,1). \end{aligned} \tag{86}$$

Moreover, given that no trade is always an option for an informed agent observing $s = s_2$ we have the inequality

$$V_0^P(x,0) \geq (1-\gamma)U(x,0) + \gamma V_1(x-z,0). \tag{87}$$

Now, suppose, by contradiction, that the equilibrium behavior for an uninformed proposer with initial portfolio x is to make zero offers. This implies

$$V_0^P(x,1/2) = (1-\gamma)U(x,1/2) + \gamma E\left[V_1\left(x,\delta'\right)\right], \tag{88}$$

where δ' denotes the period 1 belief of the agent (which may be a random variable; the expectation E is taken with respect to δ'). The fact that the value of information is zero implies that

$$V_1\left(x,\delta'\right) = \delta' V_1(x,1) + \left(1-\delta'\right)V_1(x,0).$$

Furthermore, remember that, by definition, $U(x,\delta) = \delta U(x,1) + (1-\delta)U(x,0)$ and that Bayesian updating implies $E\left[\delta'\right] = 1/2$. Then, combining (86), (87), and (88), yields

$$V_0^P(x,1/2) < \frac{1}{2}V_0^P(x,1) + \frac{1}{2}V_0^P(x,0),$$

which violates the hypothesis of zero value of information. Having ruled out the possibility that the uninformed agent with endowment x makes zero offers in equilibrium, it must be part of the equilibrium strategy to make an offer $z' \neq 0$ for the uninformed.

Case 2. Suppose it is part of the equilibrium strategy for an informed agent with initial portfolio x to make offer $z \neq 0$ at date 0 and the offer yields a strictly positive utility gain to some of the responders, but not to the proposer. If some of the responders accepting offer z and making strictly positive gains are informed agents, then it is easy to show that the informed agent making the offer would be better off making an offer that gives the responder a slightly smaller utility gain, while giving the proposer a strictly positive gain, and we are back to Case

1. Suppose instead that the only responders who accept z and make strict positive gains are uninformed agents, who update their belief to some $\delta' \in (0,1)$ and get

$$V_1\left(x+z,\delta'\right) > V_1\left(x,\delta'\right)$$

by accepting the offer. The assumption of zero value of information implies that

$$\delta' V_1\left(x+z,1\right) + \left(1-\delta'\right) V_1\left(x+z,0\right) > \delta' V_1\left(x,1\right) + \left(1-\delta'\right) V_1\left(x,0\right) \tag{89}$$

and it implies that a strategy which accepts offer z at time 0 must also be optimal for informed agents. These conditions combined imply that $V_1\left(x+z,1\right) \geq V_1\left(x,1\right)$ and $V_1\left(x+z,0\right) \geq V_1\left(x,0\right)$, with at least one strict inequality. Moreover, the fact that $\delta' \in (0,1)$ implies that the offer z was made with positive probability both by informed agents in s_1 and in s_2 (if uninformed agents are making offer z, we are back to Case 1 and we are done). Then, if $V_1\left(x+z,1\right) > V_1\left(x,1\right)$ we can perturb offer z for the informed proposer observing s_1, so that he obtains a strictly positive utility gain when making the offer. Analogously, if $V_1\left(x+z,0\right) > V_1\left(x,0\right)$ we can perturb the offer for the informed proposer observing s_2. Therefore, we have proved that at least some informed proposers must make a strictly positive utility gain and we are back to Case 1.

Step 2. Now we want to show that if offer $z \neq 0$ is part of the equilibrium strategy for a positive mass of uninformed proposers with endowment $x_{0,i}$ at time 0, then an informed agent with the same endowment can make an offer z' and obtain higher utility than following the equilibrium strategy, giving a contradiction.

First, we use the hypothesis of zero value of information to characterize the set of agents who accept offer z in equilibrium. Notice that offer z is made by a positive mass of uninformed agents. Therefore, the updated belief of an uninformed agent who receives z must be in some bounded interval $\left[\delta',\delta''\right]$ with $0 < \delta'$ and $\delta'' < 1$. Suppose z is accepted by an uninformed responder with endowment x, who updates his beliefs to some $\delta \in (0,1)$. This requires

$$V_1\left(x+z,\delta\right) \geq V_1\left(x,\delta\right). \tag{90}$$

Since $\delta \in (0,1)$, the hypothesis of zero value of information implies that accepting z is optimal for all $\delta \in [0,1]$ (this follows from an argument analogous to the one in Case 1 of Step 1). The same hypothesis implies that $V_1\left(x+z,\delta\right)$ and $V_1\left(x,\delta\right)$ are linear in δ and that (90) is equivalent to

$$\delta V_1\left(x+z,1\right) + \left(1-\delta\right) V_1\left(x+z,0\right) \geq \delta V_1\left(x,1\right) + \left(1-\delta\right) V_1\left(x,0\right).$$

These properties imply that if offer z is accepted by some responder with endowment $x_{i',0}$, then it must satisfy (90) for a responder with endowment $x_{i',0}$ and belief $1/2$, giving $V_1\left(x_{i',0}+z,1/2\right) \geq V_1\left(x_{i',0},1/2\right)$. Since the offer z is made by a proposer with endowment $x_{i,0}$ and belief $1/2$ and $z \neq 0$, it must also satisfy $V_1\left(x_{i,0}-z,1/2\right) \geq V_1\left(x_{i,0},1/2\right)$. The last two inequalities imply that the offer z is only accepted by agents with endowment $x_{-i,0}$ symmetric to $x_{i,0}$.

Suppose, without loss of generality, that $z^1 > 0$ and $z^2 < 0$. Then, conditions (12) imply that

$$V_1\left(x+z,1\right)-V_1\left(x,1\right) > V_1\left(x+z,0\right)-V_1\left(x,0\right). \tag{91}$$

That is, the agent who assigns higher probability to state S^1 is more willing to trade asset 2 for asset 1. Combining this result with the assumption of zero value of information, it follows that if we find an offer z' such that

$$V_1\left(x+z',0\right)-V_1\left(x,0\right) > 0,$$

then it follows from that

$$V_1\left(x+z',\delta\right)-V_1\left(x,\delta\right) > 0$$

for all $\delta \in [0,1]$. Therefore, irrespective of how the responder updates his belief after offer z, the offer is accepted by all agents with endowment x.

The assumption of zero value of information implies that making the offer z is optimal for informed agents with $\delta = 0$. This implies that there cannot be an offer z' such that

$$V_1\left(x-z',0\right)-V_1\left(x,0\right) > V_1\left(x-z,0\right)-V_1\left(x,0\right), \tag{92}$$

$$V_1\left(x+z',0\right)-V_1\left(x,0\right) > V_1\left(x+z,0\right)-V_1\left(x,0\right), \tag{93}$$

otherwise the proposer would make offer z', offer z' will be accepted with probability at least as large as offer z, and the proposer would strictly gain by making offer z'. Inequalities (92) and (93) imply that

$$\frac{\partial V_1\left(x-z,0\right)/\partial x^1}{\partial V_1\left(x-z,0\right)/\partial x^2} = \frac{\partial V_1\left(x+z,0\right)/\partial x^1}{\partial V_1\left(x+z,0\right)/\partial x^2}.$$

However, making the offer z is also optimal for informed agents with $\delta = 1$, so a similar reasoning implies that

$$\frac{\partial V_1\left(x-z,1\right)/\partial x^1}{\partial V_1\left(x-z,1\right)/\partial x^2} = \frac{\partial V_1\left(x+z,0\right)/\partial x^1}{\partial V_1\left(x+z,0\right)/\partial x^2}.$$

The last two equations combined lead to

$$\frac{\partial V_1\left(x-z,1\right)/\partial x^1}{\partial V_1\left(x-z,1\right)/\partial x^2} = \frac{\partial V_1\left(x-z,0\right)/\partial x^1}{\partial V_1\left(x-z,0\right)/\partial x^2},$$

which contradicts conditions (12). ■

8 Computational Appendix

This appendix describes the computational algorithms used to calculate the numerical examples in Section 5.

The agent's equilibrium strategy in period t depends on his portfolio, inherited from the previous period, x_{t-1}, his belief about the probability of state s_1, δ_{t-1}, and the distribution of portfolios and beliefs for other agents $\Gamma_t(\cdot|s)$ in that period for $s = \{s_1, s_2\}$. Notice that an individual agent cannot affect the distribution $\{\Gamma_t(\cdot|s)\}_{t=0}^{\infty}$ since agent's actions are observable only to a measure zero of agents. Therefore, each agent treats the sequence $\{\Gamma_t(\cdot|s)\}_{t=0}^{\infty}$ as given. Then the dependence on that sequence can be summarize by the calendar time t, so that the state of each agent is (x, δ, t).

At the beginning of period t, an agent has assets x_{t-1} and beliefs δ_{t-1} and chooses his optimal strategy σ_t to maximize the payoff $W(x_{t-1}, \delta_{t-1}, t)$:

$$\begin{aligned} W(x_{t-1}, \delta_{t-1}, t) &= \max_{\sigma_t} (1-\gamma) E\left\{ U(x_t(\sigma_t), \delta_t(\sigma_t)) \right| \Pr(s = s_1) = \delta_{t-1} \} \\ &\quad + \gamma E \left\{ W(x_t(\sigma_t), \delta_t(\sigma_t), t+1) \right| \Pr(s = s_1) = \delta_{t-1} \} . \end{aligned}$$

An implication of the expression above is that agent's best response strategy $\sigma^* = \{\sigma_t^*\}_{t=1}^{\infty}$ consists of a sequence of the best responses σ_t^* in a static game where agent's payoff is given by $(1-\gamma) U(\cdot, \cdot) + \gamma W(\cdot, \cdot, t+1)$. One can find equilibrium strategies of agents by the following recursive procedure:

1. Start with the initial distribution $\Gamma_0(\cdot|s)$ and compute a static Bayesian Nash equilibrium of this game with payoffs $(1-\gamma) U(\cdot, \cdot) + \gamma W(\cdot, \cdot, 1)$;

2. Use equilibrium strategies to compute the distribution in the next period, $\Gamma_1(\cdot|s)$; compute static Bayesian Nash equilibrium for the period $t = 1$;

3. Repeat the above procedure for periods $t = 2, 3, ...$

Two crucial ingredients of this procedure are: (1) finding the sequence of payoffs $\{W(\cdot, \cdot, t)\}_{t=1}^{\infty}$; and (2) finding an equilibrium in a static game with an arbitrary distribution of portfolios and beliefs $\Gamma(x, \delta|s)$ and payoffs $((1-\gamma) U + \gamma W)$.

We first describe a general procedure to compute an equilibrium. Then, we discuss some further simplifications we used for computations in Section 5.

For computational purposes, we discretize the state space and the set of offers that agents can make as follows. We fix a grid size (the step of the grid) for the offers to be h_z and for the beliefs to be h_δ. We set the bound for the size of the maximal allowed offer as $\bar{z}$. The set of allowable offers consists is given by $\mathbf{Z} = Z \times Z$, with $Z \equiv \{\pm n h_z : |n h_z| \leq \bar{z},\ n \in \mathbb{N}\}$, where $\mathbb{N}$ is a set of natural numbers. Similarly, allocations of agents take values on a set $\mathbf{X} = X \times X$, with $X \equiv \{\pm nh_z : |nh_z| \leq \bar{x},\ n \in \mathbb{N}\}$ where $\bar{x}$ is a bound on agent's allocations. Agent's beliefs take values on a set $\Delta \equiv \{0, h_\delta, 2h_\delta, ..., 1\}$.

8.1 Finding an equilibrium in a static game

The first step is to compute an equilibrium in a static, one shot game for some distribution $\Gamma : \mathbf{X} \times \Delta \to [0,1]$ and payoffs $W : \mathbf{X} \times \Delta \to \mathbb{R}$. For this purpose we adopt the algorithm of Fudenberg and Levine (1995) to our Bayesian game. This algorithm computes an approximate equilibrium for a static game, where a degree of approximation depends on a parameter κ. The algorithm has a property as $\kappa \to \infty$ the equilibrium strategies in the approximate equilibrium converge to an equilibrium in the original game[14].

1. Start with the initial guess of a probability that an offer z occurs in equilibrium if the state $s = s_1$: $\psi_0 : \mathbf{Z} \to [0,1]$, $\sum_{z \in \mathbf{Z}} \psi_0(z) = 1$, and $\psi_0(z) > 0$ for all z.

2. For any offer $z = \left(z^1, z^2\right)$ use Bayes' rule to find a posterior belief of any agent with a prior belief δ who receives an offer z:

$$\delta'(\delta, z) = \frac{\delta\psi_0(\left(z^1, z^2\right))}{\delta\psi_0((z^1, z^2)) + (1-\delta)\psi_0((z^2, z^1))}.$$

   If δ' falls outside of the grid point, we round it to the closest point on Δ. Since $\psi_0(z) > 0$ for all z, this rule is well defined.

3. Find the probability χ that an offer z is accepted in state 1. $\chi : \mathbf{Z} \to [0,1]$; $\chi(z) = \sum \Gamma(x, \delta | s_1)$ where the summation is over all $(x, \delta) \in \mathbf{X} \times \Delta$ s.t. $W(x+z, \delta'(\delta, z)) \geq W(x, \delta'(\delta, z))$.

4. Use Bayes' rule to find a posterior of the agent who makes the offer z if such an offer is accepted, δ_a, and a posterior if it is rejected, δ_r:

$$\delta_a(\delta, z) = \begin{cases} \frac{\delta\chi(\left(z^1, z^2\right))}{\delta\chi((z^1,z^2)) + (1-\delta)\chi((z^2,z^1))}, \text{ if } \delta\chi(\left(z^1, z^2\right)) + (1-\delta)\chi(\left(z^2, z^1\right)) > 0 \\ \delta, \text{ otherwise} \end{cases}$$

$$\delta_r(\delta, z) = \begin{cases} \frac{\delta(1-\chi(\left(z^1,z^2\right)))}{\delta(1-\chi((z^1,z^2))) + (1-\delta)(1-\chi((z^2,z^1)))}, \\ \text{if } \delta(1-\chi(\left(z^1, z^2\right))) + (1-\delta)(1-\chi(\left(z^2, z^1\right))) > 0 \\ \delta, \text{ otherwise.} \end{cases}$$

   If δ'' falls outside of the grid point, we round it to the closest point on Δ.

[14] See Section 3 of Fudenberg-Levine (1995) for a formal statement and a proof.

5. Find a utility $w(z;x,\delta)$ of the agent (x,δ) if he makes an offer z:

$$\begin{aligned} w(z;x,\delta) &= (\delta\chi((z^1,z^2)) + (1-\delta)\chi((z^2,z^1)))W(x-z,\delta_a(\delta,z)) \\ &\quad +(1-(\delta\chi((z^1,z^2)) + (1-\delta)\chi((z^2,z^1))))W(x,\delta_r(\delta,z)) \end{aligned}$$

If the offer $(x-z)\notin \mathbf{X}$, let $w(z;x,\delta)$ be a large negative number, $-\underline{w}$.

6. Define a strategy of an agent with (x,δ) as $\sigma_m(z;x,\delta)$:

$$\sigma_m(z;x,\delta) = \frac{\exp(\kappa w(z;x,\delta))}{\sum_{z'\in\mathbf{Z}}\exp(\kappa w(z';x,\delta))} \tag{94}$$

Here $\sigma_m(z;x,\delta)$ is the probability that agent (x,δ) makes an offer z.

7. Find a probability of each offer $\sigma_m(z) = \sum_{(x,\delta)\in\mathbf{X}\times\Delta}\sigma_m(z;x,\delta)$. If $||\sigma_m - \psi_0||$ is less than the chosen precision, finish the procedure. Otherwise, let $\psi_1 = \frac{1}{2}\psi_0 + \frac{1}{2}\sigma_m$ and go to Step 1 (for subsequent iterations use $\psi_{n+1} = \frac{n}{n+1}\psi_n + \frac{1}{n+1}\sigma_m$ and repeat the procedure until $||\sigma_m - \psi_n||$ is less than the chosen precision).

In the procedure above, (94) ensures that, for all z, $\sigma_m(z) > 0$ and, since $\psi_0(z) > 0$, $\psi_n(z) > 0$ for all z, n. This ensures that Bayes rule for updating agent's beliefs in Step 2 is well defined.

In computations in Section 5 we further reduce computational complexity by restricting out of the equilibrium beliefs for some offers. We start by considering what is the *lowest* probability that an offer z can be accepted in *any* equilibrium. This probability, $\chi^{\min}(z)$ is defined as $\chi^{\min}(z) = \sum\Gamma(x,\delta|s_1)$ where summation is over all $(x,\delta)\in\mathbf{X}\times\Delta$, s.t. $\min_{\tilde{\delta}\in[0,1]}\left\{W(x+z,\tilde{\delta}) - W(x,\tilde{\delta})\right\} > 0$. Next, we follow Steps 3-5 to compute $w(z;x,\delta)$. We define $\sigma_m(z;x,\delta) = 1$ if $z = \arg\max_{z'} w(z';x,\delta)$ and 0 otherwise and set $\chi_0(z) = \sum_{(x,\delta)\in\mathbf{X}\times\Delta}\sigma_m(z;x,\delta)$. Then we restrict the set of allowed offers to $\tilde{\mathbf{Z}} \equiv \{z\in\mathbf{Z} : \chi_0(z) > 0\}$. With these restrictions we use the iterative procedure described above. This procedure restricts all out of equilibrium beliefs to $\arg\min_{\delta'}\left\{W(x+z,\delta') - W(x,\tilde{\delta})\right\}$. Any offer in a set $\tilde{\mathbf{Z}}$ is accepted at least with a probability $\chi^{\min}$, which means that any offers in a set $\mathbf{Z}\backslash\tilde{\mathbf{Z}}$ are dominated by some offer in a set $\tilde{\mathbf{Z}}$ both on and off the equilibrium path (where out of equilibrium beliefs are constructed as the ones which imply the smallest probability of the offer being accepted).

8.2 Finding a sequence of payoffs $\{W(\cdot,\cdot,t)\}_{t=1}^{\infty}$ and an equilibrium of the dynamic game

To compute an equilibrium of a dynamic game, we truncate the game at period T. We assume that if the game has not ended before period T, it ends with probability 1 in period $T+1$.

1. Make a guess on the distribution of portfolios and beliefs $\left\{\Gamma_t^0(\cdot,\cdot|s_1)\right\}_{t=1}^T$.

2. Let $W_{T+1}^0(\cdot,\cdot) = U(\cdot,\cdot)$. Use the procedure in Section 8.1 to compute equilibrium strategies for a static game with a payoff W_{T+1}^0 and distribution Γ_T^0. Obtains functions ψ, σ_m, $\chi^{\min}$, w.

3. Compute the payoff at the beginning of the period T. For this purpose, let W^m and W^r be, respectively, the payoffs the agents who make and receive offers. Then

$$W^r(x,\delta) = \sum_{z\in\tilde{\mathbf{Z}}} \psi(z) \max\left\{W_{T+1}^0\left(x+z,\delta'(\delta,z)\right), W_{T+1}^0\left(x,\delta'(\delta,z)\right)\right\}$$

For any $\Gamma(x,\delta|s_1) > 0$ compute utility of the agent who makes an offer as

$$W^m(x,\delta) = \sum_{z\in\tilde{\mathbf{Z}}} \sigma_m(z;x,\delta) w(z;x,\delta)$$

or,

$$\begin{aligned} W^m(x,\delta) \quad = \quad & \max\{\max_{z\in\mathbf{Z}\setminus\tilde{\mathbf{Z}}}(\delta\chi^{\min}(\left(z^1,z^2\right)) + (1-\delta)\chi^{\min}(\left(z^2,z^1\right)))W(x-z,\delta_a(\delta,z)) \\ & +(1-(\delta\chi^{\min}(\left(z^1,z^2\right)) + \\ & +(1-\delta)\chi^{\min}(\left(z^2,z^1\right))))W(x,\delta_r(\delta,z)), \max_{z\in\tilde{Z}} w(z;x,\delta)\} \end{aligned}$$

The beginning of period T payoff is then $\frac{1}{2}W^m + \frac{1}{2}W^r$.

4. Set $W_T^0 = \gamma\left(\frac{1}{2}W^m + \frac{1}{2}W^r\right) + (1-\gamma)U$, and return to Step 2 until the whole sequence $\left\{W_t^0\right\}_{t=1}^T$ is computed.

5. Start with the initial distribution $\Gamma_1(\cdot,\cdot|s_1)$ and W_1^0 from Step 2 and compute the equi-

librium in a one shot game using the algorithm in Section 8.1. Compute

$$\begin{aligned}
\Gamma_2^1(\tilde{x},\tilde{\delta}|s_1) \quad = \quad & \frac{1}{2}\sum_{\substack{\{x,\delta,z:x-z=\tilde{x}\\ \delta_a(\delta,z)=\tilde{\delta}\}}} \sigma^m(z;x,\delta)(\delta\chi(\left(z^1,z^2\right))+(1-\delta)\chi(\left(z^2,z^1\right)))\Gamma_1(x,\delta|s_1) \\
& +\frac{1}{2}\sum_{\substack{\{x,\delta,z:x=\tilde{x}\\ \delta_r(\delta,z)=\tilde{\delta}\}}} \sigma^m(z;x,\delta)(1-(\delta\chi(\left(z^1,z^2\right))+(1-\delta)\chi(\left(z^2,z^1\right))))\Gamma_1(x,\delta|s_1) \\
& +\frac{1}{2}\sum_{\substack{\{x,\delta,z:\delta'(\delta,x)=\tilde{\delta},x+z=\tilde{x}\\ W(x+z,\delta'(\delta,z))\geq W(x,\delta'(\delta,z))\}}} \chi(z)\Gamma_1(x,\delta|s_1) \\
& +\frac{1}{2}\sum_{\substack{\{x,\delta,z:\delta'(\delta,x)=\tilde{\delta},x=\tilde{x}\\ W(x+z,\delta'(\delta,z))<W(x,\delta'(\delta,z))\}}} \chi(z)\Gamma_1(x,\delta|s_1)
\end{aligned}$$

The first term is the transition probabilities of all proposers whose offers are accepted. The second term is the transition probabilities of all proposers whose offers are rejected. The third term is transition probabilities of all responders who accept offers. The fourth term is the transition probabilities of all responders who reject offers. $\Gamma(x,\delta|s_2)$ can be obtained from $\Gamma(x,\delta|s_1)$ using symmetry of equilibrium.

6. Go to Step 5 until the whole sequence $\left\{\Gamma_t^1\right\}_{t=1}^{\infty}$ is computed.

7. If $||\Gamma^1-\Gamma^0||$ ($||\Gamma^{n+1}-\Gamma^n||$ in subsequent iterations) is less than chosen precision, finish the procedure. Otherwise, proceed to Step 1.

8.3 Further simplifications with exponential utility function

The procedure described above can be further simplified by assuming exponential utility function $u(x)=-\exp(-x)$ and allowing agents to have any (both positive or negative) x in all periods. In this case the strategies of any agent depend on $\left(x^1-x^2,\delta,t\right)$, which reduces the number of state variables. To see that this is the case, consider a payoff for any agent (x,δ) in period t by following some strategy σ:

$$\begin{aligned}
W(x,\delta,t)(\sigma) &= E\left\{\sum_{k=0}^{\infty}(1-\gamma)^k \begin{bmatrix} \pi(\delta_{t+k}(\sigma_{t+k}))u(x^1_{t+k}(\sigma_{t+k})) \\ +(1-\pi(\delta_{t+k}(\sigma_{t+k})))u(x^2_{t+k}(\sigma_{t+k})) \end{bmatrix} \mid \Pr(s=s_1)=\delta\right\} \\
&= E\left\{\sum_{k=0}^{\infty}(1-\gamma)^k \begin{bmatrix} \pi(\delta_{t+k}(\sigma_{t+k}))u(x^1+\sum_{m=0}^{k} z^1_{t+m}(\sigma_{t+m})) \\ +(1-\pi(\delta_{t+k}(\sigma_{t+k})))u(x^2+\sum_{m=0}^{k} z^2_{t+m}(\sigma_{t+m})) \end{bmatrix} \mid \Pr(s=s_1)=\delta\right\} \\
&= \exp(-x^2) * \\
& \quad E\left\{\sum_{k=0}^{\infty}(1-\gamma)^k \begin{bmatrix} \pi(\delta_{t+k}(\sigma_{t+k}))u(\big(x^1-x^2\big) \\ +\sum_{m=0}^{k} z^1_{t+m}(\sigma_{t+m})) \\ +(1-\pi(\delta_{t+k}(\sigma_{t+k})))u(\sum_{m=0}^{k} z^2_{t+m}(\sigma_{t+m})) \end{bmatrix} \mid \Pr(s=s_1)=\delta\right\}
\end{aligned}$$

Consider any two strategies, σ' and σ'', s.t. $W(x,\delta,t)(\sigma') \geq W(x,\delta,t)(\sigma'')$ for some (x,δ). Since we do not impose bounds on asset holdings x_t, the same strategies σ' and σ'' are feasible for all agents. But then the last expression implies that $W(\tilde{x},\delta,t)(\sigma') \geq W(\tilde{x},\delta,t)(\sigma'')$ for all $\tilde{x}$ s.t. $\tilde{x}^1 - \tilde{x}^2 = x^1 - x^2$.

References

[1] Amador, Manuel and Pierre Olivier Weill (2007). "Learning from Private and Public Observation of Other's Actions." Mimeo, UCLA.

[2] Amador, Manuel and Pierre Olivier Weill (2008). "Learning from Prices: Public Communication and Welfare." Mimeo, UCLA.

[3] Billingsley, Patrick (1995). *Probability and Measure.* Wiley Series in Probability and Mathematical Statistics.

[4] Blouin, Max R., and Roberto Serrano (2001). "A Decentralized Market with Common Values Uncertainty: Non-Steady States," *Review of Economic Studies* 68: 323–346.

[5] Brunnermeier, Markus (2001). *Asset Pricing under Asymmetric Information.* Oxford University Press.

[6] Cho, In-Koo and David M. Kreps (1987). "Signalling Games and Stable Equilibria." *Quarterly Journal of Economics*, 102(2): 179-221.

[7] Duffie, Darrell, Nicolae Garleanu and Lasse Heje Pedersen (2005). "Over-the-Counter Markets." *Econometrica* 73: 1815–1847.

[8] Duffie, Darrell, Nicolae Garleanu and Lasse Heje Pedersen (2007). "Valuation in Over-the-Counter Markets." *Review of Financial Studies*, forthcoming.

[9] Duffie, Darrell, Gaston Giroux, and Gustavo Manso (2009). "Information Percolation," *American Economic Journal: Microeconomics*, forthcoming.

[10] Duffie, Darrell, Semyon Malamud and Gustavo Manso (2009a). "Information Percolation with Equilibrium Search Dynamics." *Econometrica*, forthcoming.

[11] Duffie, Darrell, Semyon Malamud and Gustavo Manso (2009b). "The Relative Contributions of Private Information Sharing and Public Information Releases to Information Aggregation." J*ournal of Economic Theory*, forthcoming.

[12] Duffie, Darrell, and Gustavo Manso, (2007). "Information Percolation in Large Markets," *American Economic Review, Papers and Proceedings*, 97: 203-209.

[13] Fudenberg, Drew, and David Levine (1995). "Consistency and Cautious Fictitious Play." *Journal of Economic Dynamics and Control* 19: 1065-1089.

[14] Gale, Douglas (1987). "Limit Theorems for Markets with Sequential Bargaining." *Journal of Economic Theory* 43: 20–54.

[15] Gale, Douglas (2000). *Strategic Foundations of General Equilibrium: Dynamic Matching and Bargaining Games.* Cambridge University Press.

[16] Grossman, Sanford (1989). *The Informational Role of Prices.* MIT Press.

[17] Lagos, Ricardo (2007), "Asset Prices and Liquidity in an Exchange Economy," working paper, NYU.

[18] Lagos, Ricardo and Guillaume Rocheteau (2009). "Liquidity in Asset Markets with Search Frictions," *Econometrica*, 77(2) (March 2009): 403-426.

[19] Lagos, Ricardo, and Guillaume Rocheteau and Pierre Olivier Weill, (2007). "Crashes and Recoveries in Illiquid Markets," working paper.

[20] Milgrom, Paul R., and Nancy Stokey (1982). "Information, Trade and Common Knowledge." *Journal of Economic Theory* 26: 17–27.

[21] Ostrovsky, Michael (2009). "Information Aggregation in Dynamic Markets with Strategic Traders," Working paper.

[22] Rocheteau, Guillaume (2008). "A Monetary Approach to Asset Liquidity," Working paper.

[23] Wolinsky, Asher (1990), "Information Revelation in a Market with Pairwise Meetings," *Econometrica* 58: 1–23.

[24] Vayanos, Dimitri (1998). "Transaction Costs and Asset Prices: A Dynamic Equilibrium Model." *Review of Financial Studies* 11, 1–58.

[25] Vayanos, Dimitri and Pierre Olivier Weill (2007). "A Search-Based Theory of the On-the-Run Phenomenon." *Journal of Finance*, forthcoming.

[26] Weill, Pierre Olivier (2007). "Liquidity Premia in Dynamic Bargaining Markets." *Journal of Economic Theory*, forthcoming.

[27] Williams, David (1991). *Probability with Martingales.* Cambridge University Press