14.471/Fall 2003

Solutions to Problem Set 1

Question 1:

(a) Consumer i (i = 1,2,...,N) of type 1 solves the following constrained utility maximization problem:

$$\max_{x_1^i, g_1^i} \quad \log(x_1^i) + \log(g_1^i + G^{-i})$$
$$s.t. \quad x_1^i + g_1^i = 1$$

Solving the budget constraint for g_1^i, we can get the following univariate maximization problem and first order condition:

$$\max_{g_1^i} \quad \log(1 - g_1^i) + \log(g_1^i + G^{-i})$$

FOC: $1 - g_1^i = g_1^i + G^{-i}$

Similarly, a consumer j (j = 1,2,...,N) of type 2 has the following utility maximization problem and first order condition:

$$\max_{g_2^j} \quad \log(\tfrac{1}{2} - g_2^j) + \beta \log(g_2^j + G^{-j})$$

FOC: $\frac{1}{2} - g_2^j = \frac{1}{\beta}(g_2^j + G^{-j})$

To find the symmetric Nash equilibrium, set $g_1^i = g_1 \; \forall i$ and $g_2^j = g_2 \; \forall j$ in the two types' first order conditions which yields the following two equations in two unknowns.

$$1 - g_1 = Ng_1 + Ng_2$$
$$\tfrac{1}{2} - g_2 = \tfrac{1}{\beta}(Ng_1 + Ng_2)$$

Rearranging the first equation, we get:

$$g_1 = \tfrac{1}{N+1}(1 - Ng_2)$$

And substituting this in for g_1 in the second equation gives:

$g_2 = \frac{1}{2}\frac{\beta N + \beta - 2N}{\beta N + \beta + N} \leq 0 \Rightarrow \; g_2 = 0$ (i.e. a corner solution, since g_2 is constrained to be ≥ 0)

Knowing that the amount of the public good supplied by type 2 individuals is zero, we can solve for the amount supplied by an individual of type1 and for the aggregate amount of public good that will be privately provided:

$$g_1 = \tfrac{1}{N+1} \quad and \quad G = Ng_1 = \tfrac{N}{N+1}$$

(b) We now assume that the government will provide the public good and finance its production by levying a constant lump sum tax on all 2N individuals. Furthermore, we will assume that the government has effectively seized the factories that can produce the public good, so that individuals who think that the government has not provided enough of the public good cannot provide more of it from their own private resources.

The government maximizes a utilitarian social welfare function when choosing the optimal amount of public good to provide (and, hence, the optimal lump sum tax it will levy):

$$\max_{T,G} \quad N\log(1-T) + N\log(\tfrac{1}{2}-T) + N\log(G) + N\beta\log(G)$$
$$s.t. \quad G = 2NT$$

Substituting 2NT in for G, we obtain the following first order condition:

$$\text{FOC:} \quad \tfrac{1}{1-T} + \tfrac{1}{1/2-T} = \tfrac{(1+\beta)}{T}$$

Solving for T, the optimal lump sum tax (after lots of algebra):

$$T = \tfrac{1}{2(6+2\beta)}\left(6+3\beta-\sqrt{(12+4\beta+\beta^2)}\right)$$
$$\Rightarrow G = \tfrac{1}{(6+2\beta)}\left(6+3\beta-\sqrt{(12+4\beta+\beta^2)}\right)N$$

Note also that as the population (and aggregate endowment) grows the amount of public good provided goes to infinity. The optimal lump sum tax is independent of the size of the population. The socially optimal amount of the public good is greater than the amount provided in the Nash equilibrium of part (a) if (1) $N > 1$ or (2) $N = 1$ and $\beta > 1/3$. (Note that $G|_{\beta=1/3,N=1} = 1/2$.)

Question 2:

If an individual with endowment x (x = 1,2, or 3) were given reign to decide the amount of aggregate public good, she would solve the following constrained utility maximization problem:

$$\max_{T,G} \quad \log(x-T) + \log(G)$$
$$s.t. \quad \tfrac{3NT}{2} = G$$

The solution to this problem is:

$$T = \tfrac{x}{2} \quad and \quad G = \tfrac{3x}{4}N$$

The median voter model result will apply in this setting. Individuals with endowment x = 2 will be the median voters (i.e. marginal or decisive) voters since their utility-maximizing amount of the public good is in the middle of the range of optimal amounts (and because of some other assumptions about the voting process). The median voters' utility-maximizing amount of the public good will be the outcome of the "political process" so that:

$$T = 1 \quad and \quad G = \tfrac{3}{2}N$$

The amount of the public good chosen does depend on N. This is because N affects the "price" of the public good that the median voter faces. The more people there are, the cheaper is each unit of the public good for the individual. Because the median voter has Cobb-Douglas preferences, she will spend a constant proportion of her income on the public good. Therefore, as N increases her optimal amount of G increases (as the price of G has fallen and her income has remained the same).

Question 3:

The consumer solves the following constrained maximization problem:

$$\max_{x_1,x_2,x_3}, \quad \beta_1\log(x_1-\alpha_1) + \beta_2\log(x_2-\alpha_2) + \beta_3\log(x_3)$$
$$s.t. \quad q_1x_1 + q_2x_2 + x_3 = y$$

To simplify matters later on, I will assume that $\beta_1 + \beta_2 + \beta_3 = 1$. (This just involves dividing the utility fcn by a normalizing constant.) Setting up a Langrangian we can derive the following FOCs:

$$\frac{\partial L}{\partial x_1} = \frac{\beta_1}{x_1-\alpha_1} - \lambda q_1 \le 0 \quad (= 0 \text{ for } x_1 > 0)$$
$$\frac{\partial L}{\partial x_2} = \frac{\beta_2}{x_2-\alpha_2} - \lambda q_2 \le 0 \quad (= 0 \text{ for } x_2 > 0)$$
$$\frac{\partial L}{\partial x_3} = \frac{\beta_3}{x_3} - \lambda q_3 \le 0 \quad (= 0 \text{ for } x_3 > 0)$$
$$\frac{\partial L}{\partial \lambda} = y - (q_1 x_1 + q_2 x_2 + x_3) \ge 0$$

The Marshallian demands are given by:

$$x_i = \alpha_i + \frac{\beta_i(y-\alpha_1 q_1-\alpha_2 q_2)}{q_i} \quad i = 1,2,3$$

Plugging the Marshallian demands into the utility function and simplifying gives us the indirect utility function:

$$v(q,y) = \beta_1\log(\beta_1) + \beta_2\log(\beta_2) + \beta_3\log(\beta_3) - \beta_1\log(q_1) - \beta_2\log(q_2) + (\beta_1+\beta_2+\beta_3)\log(y -$$

To find the expenditure function, solve the indirect utility function for y = e(q,u), where u = v(q,y).

$$u + \beta_1\log(q_1) + \beta_2\log(q_2) - \{\beta_1\log(\beta_1) + \beta_2\log(\beta_2) + \beta_3\log(\beta_3)\} = \log(e(q,u) - \alpha_1 q_1 - \alpha_2 q_2)$$

$$\Rightarrow e(q,u) = \alpha_1 q_1 + \alpha_2 q_2 + \exp\{u + \beta_1\log(q_1) + \beta_2\log(q_2) + c\}$$
where c = $-\{\beta_1\log(\beta_1) + \beta_2\log(\beta_2) + \beta_3\log(\beta_3)\}$

$$\Rightarrow e(q,u) = \alpha_1 q_1 + \alpha_2 q_2 + e^u q_1^{\beta_1} q_2^{\beta_2} e^c$$

(b) Let $\alpha_1 = \alpha_2 = 0.5$ and $\beta_1 = \beta_2 = 0.4$. Prices change from $(q_1,q_2) = (1,1)$ to $(q_1,q_2) = (2,1.5)$. Income remains at $y = 5$.

$$-CV = e(q^1,u^0) - e(q^0,u^0) = .5 * 2 + .5 * 1.5 + e^{u^0} * 2^{.4} * 1.5^{.4} * e^{1.0549} - 5$$

$$u^0 = v(q^0,y) = 0.33137$$

So, $CV = -2.957$

For normal goods (i.e. goods with positive income elasticities) CV will be greater than EV in absolute value when there is a price increase. Draw a two good diagram of the Hicksian and Marshallian demands with a price increase to convince yourselves.

(c) Revenue will equal the tax rates on each good multiplied by the Marshallian demand for each good at the new prices. Here the tax rates are given by $q^1 - q^0$.

The Marshallian demands for goods 1 and 2 at q^1 are:
$x_1 = \frac{23}{20}$ and $x_2 = \frac{41}{30}$

Revenue= $(2-1)*(\frac{23}{20}) + (1.5-1)*(\frac{41}{30}) = \frac{11}{6}$.

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1a) Type 1s maximize $\ln x_{1i} + \ln(g_{1i} + G^{-i})$ s.t. $x_{1i} + g_{1i} = 1$

$\Rightarrow \ln x_{1i} + \ln(1 - x_{1i} + G^{-i})$

FOC: $\frac{1}{x_{1i}} = \frac{1}{1 - x_{1i} + G^{-i}} \Rightarrow x_{1i} = 1 - x_{1i} + G^{-i} \Rightarrow x_{1i} = \frac{1 + G^{-i}}{2}$

$\Rightarrow g_{1i} = 1 - \frac{1}{2} - \frac{G^{-i}}{2} = \frac{1 - G^{-i}}{2} \Rightarrow \boxed{1 - g_{1i} = g_{1i} + G^{-i}}$

Type 2s max $\ln x_{2i} + \beta \ln(g_{2i} + G^{-i})$ s.t. $x_{2i} + g_{2i} = \frac{1}{2}$

$\Rightarrow \ln x_{2i} + \beta \ln(\frac{1}{2} - x_{2i} + G^{-i})$

FOC: $\frac{1}{x_{2i}} = \frac{\beta}{\frac{1}{2} - x_{2i} + G^{-i}} \Rightarrow \beta x_{2i} = \frac{1}{2} - x_{2i} + G^{-i} \Rightarrow x_{2i} = \frac{\frac{1}{2} + G^{-i}}{1 + \beta}$

$\Rightarrow g_{2i} = \frac{1}{2} - \frac{\frac{1}{2} + G^{-i}}{1 + \beta} \Rightarrow \boxed{\frac{1}{2} - g_{2i} = \frac{1}{\beta}(g_{2i} + G^{-i})}$

$\Rightarrow$ now let $G^{-i} = Ng_1 + Ng_2$ and generalize $\forall i$:

$1 - g_1 = g_1 + (N-1)g_1 + Ng_2 = Ng_1 + Ng_2$

$\Rightarrow 1 - g_1 = Ng_1 + Ng_2 \Rightarrow g_1 = \frac{1}{N+1}(1 - Ng_2)$

$\frac{1}{2} - g_2 = \frac{1}{\beta}(Ng_1 + Ng_2)$

$\Rightarrow \frac{1}{2} - g_2 = \frac{1}{\beta}\left(\frac{N}{N+1}(1 - Ng_2) + Ng_2\right) = \frac{1}{\beta}\left(\frac{N}{N+1} - \frac{N^2}{N+1} g_2 + Ng^2\right)$

$\Rightarrow \frac{1}{2} - \frac{N}{\beta(N+1)} = g_2 - \frac{N^2}{\beta(N+1)} g_2 + \frac{N}{\beta} g_2 = \left(\frac{\beta N + \beta - N^2 + N^2 + N}{\beta(N+1)}\right) g_2$

$\Rightarrow \frac{-2N + \beta N + \beta}{\cancel{2\beta(N+1)}} = \left(\frac{\beta N + \beta + N}{\cancel{\beta(N+1)}}\right) g_2$

$\Rightarrow g_2 = \frac{-2N + \beta N + \beta}{2(\beta N + \beta + N)} \Rightarrow g_2 = 0 \Rightarrow g_1 = \frac{1}{N+1}$

$G = Ng_1 + Ng_2 = \frac{N}{N+1} + N \cdot 0 = \frac{N}{N+1}$

1b. $\max_{T,G} N\ln(1-T) + N\ln(\tfrac{1}{2}-T) + N\ln G + \beta N \ln G$ s.t. $G = 2NT$

$$\Rightarrow \max_{T} N\ln(1-T) + N\ln(\tfrac{1}{2}-T) + (1+\beta)N\ln 2NT$$

FOC: $$\frac{N}{1-T} + \frac{N}{\frac{1}{2}-T} = \frac{(1+\beta)N}{T} \Rightarrow \frac{(\frac{3}{2}-2T)N}{(1-T)(\frac{1}{2}-T)} = \frac{(1+\beta)N}{T}$$

$$\Rightarrow \frac{3-4T}{1-3T+2T^2} = \frac{1+\beta}{T} \Rightarrow 3T - 4T^2 = 1+\beta - (3+3\beta)T + (2+2\beta)T^2$$

$$\Rightarrow 0 = (6+2\beta)T^2 - (6+3\beta)T + (1+\beta)$$

$$\Rightarrow T = \frac{6+3\beta - \sqrt{(6+3\beta)^2 - 4(1+\beta)(6+2\beta)}}{2(6+2\beta)}$$

$$= \frac{6+3\beta - \sqrt{12+4\beta+\beta^2}}{2(6+2\beta)}$$

$$G = 2NT = \frac{N}{6+2\beta}\left[6+3\beta - \sqrt{12+4\beta+\beta^2}\right]$$

2. $\max_{T,G} \ln(x-T) + \ln G$ s.t. $\frac{3NT}{2} = G$

$$\Rightarrow \ln(x-T) + \ln\frac{3NT}{2}$$

FOC: $$\frac{1}{x-T} = \frac{1}{T} \Rightarrow T = x-T \Rightarrow x = 2T \Rightarrow T = \frac{x}{2}$$

$$G = \frac{3N}{2}\cdot\frac{x}{2} = \frac{3Nx}{4}$$

3a. $\max_{x_1,x_2,x_3} \beta_1 \ln(x_1-\alpha_1)+\beta_2\ln(x_2-\alpha_2)+\beta_3\ln x_3$ s.t. $q_1x_1+q_2x_2+q_3x_3 = y$

$$\mathcal{L} = \beta_1\ln(x_1-\alpha_1)+\beta_2\ln(x_2-\alpha_2)+\beta_3\ln x_3+\lambda(y-q_1x_1-q_2x_2-q_3x_3)$$

FOCs: $\frac{\partial\mathcal{L}}{\partial x_1}: \frac{\beta_1}{x_1-\alpha_1}-\lambda q_1 = 0 \qquad \frac{\partial\mathcal{L}}{\partial x_2}: \frac{\beta_2}{x_2-\alpha_2}-\lambda q_2 = 0$

$\frac{\partial\mathcal{L}}{\partial x_3}: \frac{\beta_3}{x_3}-\lambda q_3 = 0 \qquad \frac{\partial\mathcal{L}}{\partial \lambda}: y-q_1x_1-q_2x_2-q_3x_3 = 0$

$\forall i,\ x_i = \frac{\beta_i}{\lambda q_i}+\alpha_i \Rightarrow$ what is $\frac{1}{\lambda}$?

Trick: $\beta_1+\beta_2+\beta_3 = 1$ (utility function doesn't change if we normalize the coefficients)

$$\Rightarrow \frac{1}{\lambda} = \frac{\beta_1+\beta_2+\beta_3}{\lambda} = \frac{\beta_1}{\lambda}+\frac{\beta_2}{\lambda}+\frac{\beta_3}{\lambda}$$

$$= q_1x_1 - q_1\alpha_1 + q_2x_2 - q_2\alpha_2 + x_3$$

$$= \underbrace{q_1x_1+q_2x_2+x_3}_{=y} - q_1\alpha_1 - q_2\alpha_2$$

$$= y - q_1\alpha_1 - q_2\alpha_2$$

$$\Rightarrow \boxed{x_i = \frac{\beta_i}{q_i}(y-q_1\alpha_1-q_2\alpha_2)+\alpha_i}$$ The Marshallians

$$\Rightarrow u = \beta_1\ln\beta_1+\beta_2\ln\beta_2+\beta_3\ln\beta_3-\beta_1\ln q_1-\beta_2\ln q_2+\overbrace{(\beta_1+\beta_2+\beta_3)}^{=1}\ln(y-q_1\alpha_1-q_2\alpha_2)$$

$$\boxed{u = v(q,y) = \sum\beta_i\ln\beta_i - \sum\beta_i\ln q_i + \ln(y-\alpha_1q_1-\alpha_2q_2)}$$ indirect utility fn

$e(q,u)$

$$\Rightarrow \ln(e(q,u)-\alpha_1q_1-\alpha_2q_2) = u+\sum\beta_i\ln q_i-\sum\beta_i\ln\beta_i$$

$$\Rightarrow e(q,u) = \alpha_1q_1+\alpha_2q_2+\exp\Big[u+\sum\beta_i\ln q_i\underbrace{-\sum\beta_i\ln\beta_i}_{=c=1.05}\Big]$$

$$\boxed{e(q,u) = \alpha_1q_1+\alpha_2q_2+q_1^{\beta_1}q_2^{\beta_2}e^{u+c}}$$ expenditure fn

3b. $-CV = e(q_b^1, u^0) - \underbrace{e(q^0, u^0)}_{=y}$

$$= .5 \cdot 2 + .5 \cdot 1.5 + e^{u_0} \cdot 2^{.4} \cdot 1.5^{.4} \cdot e^{1.05} - 5$$

$u^0 = .83 = v(q_b^0, y)$

$\Rightarrow CV = -2.96$