A Theory of House Allocation and Exchange Mechanisms*

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Abstract

We study the allocation and exchange of indivisible objects without monetary transfers. In market design literature, some problems that fall in this category are the house allocation problem with and without existing tenants, and the kidney exchange problem. We introduce a new class of direct mechanisms that we call "trading cycles with brokers and owners," and show that (i) each mechanism in the class is coalitional strategy-proof and Pareto-efficient, and (ii) each coalitional strategy-proof and Pareto-efficient direct mechanism is in the class. As corollaries, we obtain new characterizations in the aforementioned market design problems.

Keywords: Mechanism design, coalitional strategy-proofness, Pareto-efficiency, matching, house allocation.

JEL classification: C78, D78

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1 Introduction

The theory and practical applications involving the allocation and exchange of indivisible resources without monetary transfers have recently been attracting attention of economists. Market designers have tailored new models and mechanisms to solve real-life problems such as the allocation of students to on-campus dormitory rooms at US colleges (cf. Abdulkadiroğlu and Sönmez 1999) and exchanges of live donor kidney transplants (cf. Roth, Sönmez, and Ünver 2004).

There are common features of these real-life problems. There is a group of agents each of whom would like to consume an indivisible object to which we will refer to as a house using the terminology coined by Shapley and Scarf (1974). Moreover, there is a group of houses to be distributed according to the agents' strict preferences over the houses. We will refer to such problems as house allocation and exchange problems. We study direct revelation mechanisms, that is, agents reveal their preferences over houses, and the mechanism assigns a house to each agent.

The direct mechanisms studied in the literature have two essential properties: Pareto-efficiency and coalitional strategy-proofness. Coalitional strategy-proofness means that no group of agents can jointly manipulate so that all of them weakly benefit from this manipulation, while at least one in the group strictly benefits. In this domain of problems, coalitional strategy-proofness has a second nice interpretation and is equivalent to individual strategy-proofness and non-bossiness, two important non-cooperative properties (cf. Pápai 2000). Individual strategy-proofness is dominant strategy incentive compatibility property, i.e., no agent can strictly benefit by manipulation of the mechanism. Such mechanisms are not only non-manipulable but also impose minimal computational costs on the participants and do not discriminate agents based on their ability to strategize and their access to information (cf. Vickrey 1961, Dasgupta, Hammond, and Maskin 1979, and Pathak and Sönmez 2008). Non-bossiness means that if an agent cannot change his own assignment under the mechanism by manipulation, then he can change nobody else's assignment, i.e., the mechanism outcome remains unchanged. Non-bossiness is a robustness property, which is often taken for granted in the literature. The message relayed by an individual is used to determine a non-bossy mechanism's outcome only to the extent to determine the individual's own assignment (cf. Satterthwaite and Sonnenschein 1981).

We introduce a new class of direct mechanisms that we call trading cycles with brokers and owners, and show that (i) each mechanism in the class is coalitionally strategy-proof and Pareto-efficient, and (ii) each coalitionally strategy-proof and Pareto-efficient direct mechanism can be implemented through a mechanism from the class. Thus, we characterize the full class of relevant direct mechanisms, and lay down the structure of the house allocation and exchange problem. The trading-cycles-with-brokers-and-owners mechanisms can be used to solve practical design problems that were beyond the reach of the previously known mechanisms.

A trading-cycles-with-brokers-and-owners algorithm matches houses and agents in a sequence of rounds. At each round some agents and houses are matched and removed from the problem. At the beginning of the round, each previously unmatched house is controlled by an unmatched agent. We distinguish two forms of control over a house which we call ownership and brokerage (at any round, there is at most one broker and one brokered house). Each house points to the agent that controls it, and each agent points to his most preferred unmatched house. The only exception is the broker (if there is one) who points to his most preferred unmatched house other than the brokered house. In the resultant directed graph, there exists at least one exchange cycle. Each agent in each exchange cycle is matched with the house he points to.

The allocation of control rights in each round is fully determined by how agents and houses were matched prior to that round. The above-described procedure takes as given the mapping from partial matchings to control rights. Each such mapping that satisfies certain compatibility conditions determines a mechanism in our class.

The above class of mechanisms is built on the top-trading cycles idea attributed to David Gale by Shapley and Scarf (1974), and developed by Abdulkadiroğlu and Sönmez (1999) and Pápai (2000). The subclass of our mechanisms without brokers was introduced by Pápai (2000); it is the largest class of coalitionally strategy-proof and Pareto-efficient mechanisms previously known.

Against this background, our main innovation lies in introducing the brokerage control rights. Previously only ownership control rights were studied in the context of house allocation and exchange. Recognizing the role of brokers in house allocation and exchange is crucial to obtaining the entire class of coalitionally strategy-proof and Pareto-efficient mechanisms. The introduction of brokers is also useful in some design problems.

As an example of a mechanism design problem in which brokerage rights are useful, consider a manager who assigns n tasks $t_1, ..., t_n$ to n employees $w_1, ..., w_n$ with strict preferences over the tasks. The manager wants the allocation to be Pareto-efficient with regard to the employees' preferences. Within this constraint, she would like to avoid assigning task t_1 to employee w_1. She wants to use a coalitionally strategy-proof direct mechanism, because she does not know employees' preferences. The only way to do it using the previously known mechanisms is to endow employees $w_2, ..., w_n$ with the tasks, let them find the Pareto-efficient allocation through a top-trading cycles procedure, such as Pápai's (2000) hierarchical exchange, and then allocate the remaining task to employee w_1. Ex ante each such procedure is unfair to the employee w_1. Using a trading-cycles-with-brokers-and-owners mechanism, the manager can achieve her objective without the extreme discrimination of the employee w_1. She makes w_1 the broker of t_1, allocates the remaining tasks among $w_2, ..., w_n$ (for instance she may make w_i the owner of t_i, $i = 2, ..., n$), and runs trading cycles with brokers and owners.

In our main result, we assume that all houses are social endowments, and hence there are no exogenous constraints on the allocation of control rights (cf. Hylland and Zeckhauser 1979). For instance, at some universities, the dormitory rooms are treated as social endowments. At other universities however, some students, such as sophomores, have the right to stay in the room they lived in the preceding year. In kidney exchange, patients (interpreted as agents) come with a paired-donor (interpreted as a house) and have to be matched with at least their paired-donor. We derive corollaries of our main result for problems in which some houses are private endowments of agents and the participation in the mechanism has to be individually rational.

There are many studies that characterize desirable properties of house allocation and exchange through variants of top-trading cycles mechanisms. The most general class of mechanisms in the literature prior to our study was constructed by Pápai (2000). Her class characterizes coalitional strategy-proofness and Pareto-efficiency together with an additional property which is referred to as reallocation-proofness. A mechanism is reallocation-proof if there does not exist a profile of preferences, a pair of agents and a pair of preference misrepresentations such that (i) if both of them misrepresent their preferences, both of them weakly gain and one of them strictly gain by swapping their assignments, and (ii) if only one of them misrepresents his preferences, he cannot change his assignment. She also notes that the stronger property without condition (ii) conflicts with coalitional strategy-proofness and Pareto-efficiency. We do not use reallocation-proofness in our result.[1]

In matching and house allocation and exchange literature, the standard modeling approach has

[1] Ma (1994), Svensson (1999), Ergin (2000), Miyagawa (2002), Ehlers, Klaus, and Pápai (2002), Ehlers and Klaus (2004), Kesten (2004), Sönmez and Ünver (2006), Ehlers and Klaus (2007), and Velez (2008) characterize subclasses of coalitionally strategy-proof, Pareto-efficient, and reallocation-proof mechanisms.

been to use strict preferences instead of the full preference domain. Participants are frequently allowed to submit only strict preference orderings to real-life direct mechanisms in various markets, such as dormitory room allocation, school choice, matching of interns and hospitals. As Ehlers (2002) shows "one cannot go much beyond strict preferences if one insists on efficiency and coalitional strategy-proofness." He characterizes coalitionally strategy-proof and Pareto-efficient mechanisms in the maximal subset of full preference domain such that such a mechanism exists. The full preference domain gives rise to an impossibility result, i.e., when agents can be indifferent among houses, there exists no mechanism that is coalitionally strategy-proof and Pareto-efficient. Under strict preferences, his class of mechanisms is a subclass of ours, and substantially different from the general class.[2]

The study of strategy-proof and Pareto-efficient mechanisms has a long tradition. Gibbard (1973) and Satterthwaite (1975) have shown under minor restrictions that all strategy-proof voting rules are dictatorial. Satterthwaite and Sonnenschein (1981) extended this result to public good economies with production, and Zhou (1991) extended it to pure public good economies. In social choice models, Dasgupta, Hammond and Maskin (1979) have proved that every Pareto-efficient and strategy-proof social choice rule is dictatorial. In exchange economies, Barberà and Jackson (1995) showed that strategy-proof mechanisms are Pareto-inefficient.

Even with additional structure, it has been difficult to characterize Pareto efficient and strategy-proof mechanisms that are non-dictatorial. Such characterizations have been obtained by Green and Laffont (1977) in decision problems with monetary transfers and quasi-linear utilities (cf. Vickrey 1961, Clarke 1971, Groves 1973) and by Barberà, Gül, and Stacchetti (1993) in voting problems with single-peaked preferences (cf. Moulin 1980 and Sprumont 1991).[3]

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[2]See Bogomolnaia, Deb, and Ehlers (2005) for another characterization with indifferences.

[3]Sönmez (1999) studies generalized matching problems in which each agent is endowed with a good. The class of such problems non-trivially intersects with the class of house allocation and exchange problems studied in this paper. He shows that (i) there exists a Pareto-efficient, strategy-proof, and individually rational mechanism if and only if the core is nonempty and agents are indifferent between all core allocations. He also shows that any such mechanism is coalitionally strategy-proof (cf. Shapley and Scarf 1974, Roth and Postlewaite 1977, Roth 1982, and Ma 1994).

2 The Model

Let I be a set of **agents** and H be a set of **houses.** We use letters i, j, k to refer to agents and h, g, f to refer to houses. Each agent i has a **strict preference relation** over H, denoted by $\succ_i$.[4] Let $\mathbf{P}_i$ be the set of strict preference relations for agent i. Let $\mathbf{P}_J \equiv \Pi_{i\in J}\mathbf{P}_i$ be Cartesian product of the set of preferences for all agents in any $J \subseteq I$, and let $\mathbf{P} \equiv \mathbf{P}_I$. Let $\succ = (\succ_i)_{i\in I} \in \mathbf{P}$ be a **preference profile**. For all $\succ \in \mathbf{P}$ and all $J \subseteq I$, let $\succ_J = (\succ_i)_{i\in J}$ be the restriction of $\succ$ to J. A **house allocation problem** is denoted as a triple $\langle I, H, \succ\rangle$. We will assume that $|H| \geq |I|$ so that each agent is allocated a house. Each house allocation problem $\langle I, H, \succ\rangle$ that violates this assumption might be embedded in an augmented problem $\langle I, H', \succ'\rangle$ in which $H' \supseteq H$, the restriction of $\succ'$to houses in H coincides with $\succ$, and $|H'| \geq |I|$.

An outcome of a house allocation problem is a *matching*. To define a matching, let us start with a more general concept that we will use frequently. A **submatching** is an allocation of a subset of houses to a subset of agents, such that no two different agents get the same house. Formally, a submatching is a one-to-one and onto function $\sigma : J \to G$ such that $J \subseteq I$, $G \subseteq H$ and $|J| = |G|$. Let $\sigma(i)$ be the assignment of agent i under σ. Let $\mathcal{S}$ be the set of submatchings. For each $\sigma \in \mathcal{S}$ with $\sigma : J \to G$, let $I_\sigma \equiv J$ be the set of agents and $H_\sigma \equiv G$ be the set of houses matched under σ. Whenever it is convenient, we represent a submatching $\sigma \in \mathcal{S}$ as a set of matches,

$$\sigma = \{(i, \sigma(i))\}_{i\in I_\sigma}.$$

For all $h \in H$, let $\mathcal{S}_{-h} \subset \mathcal{S}$ be the set of submatchings $\sigma \in \mathcal{S}$ with $h \notin H_\sigma$, i.e. the set of submatchings at which h is unmatched. For each $h \in H_\sigma$ and $\sigma \in \mathcal{S}_{-h}$, $\sigma^{-1}(h) \in I_\sigma$ is the agent that got house h under σ, that is, $\sigma(i) = h$.

A **matching** is a submatching that matches all agents in I. Formally, a matching is a submatching $\mu \in \mathcal{S}$ such that $I_\mu = I$. Let $\mathcal{M} \subset \mathcal{S}$ be the set of matchings.

A **(direct) mechanism** is a systematic procedure that assigns a matching for each problem. Throughout the paper, we fix I and H, and thus, a problem is identified with its preference profile. Therefore, formally a mechanism is a function $\varphi : \mathbf{P} \longrightarrow \mathcal{M}$.

[4] Let $\succeq_i$ be the induced weak preference relation, that is for any $g, h \in H$, $g \succeq_i h \iff g = h$ or $g \succ_i h$. A weak preference relation is a linear order on H, i.e. a binary relation on H that is antisymmetric, transitive, complete, and reflexive.

3 Coalitional Strategy-Proofness and Pareto Efficiency

In this section, we introduce essential properties of house allocation mechanisms.

A matching is **Pareto-efficient**, if there is no matching that makes everybody weakly better off, and at least one agent strictly better off. That is, a matching $\mu \in \mathcal{M}$ is Pareto-efficient if there exists no matching $\nu \in \mathcal{M}$ such that for all $i \in I$, $\nu(i) \succeq_i \mu(i)$, and for some $i \in I$, $\nu(i) \succ_i \mu(i)$. A mechanism is **Pareto-efficient**, if it finds a Pareto-efficient matching for every problem.

A mechanism is *coalitionally strategy-proof* if there is no group of agents that can misstate their preferences in a way such that each one in the group gets a weakly better house, and at least one agent in the group gets a strictly better house. Formally, a mechanism φ is **coalitionally strategy-proof** if for all $\succ \in \mathbf{P}$, there exists no $J \subseteq I$ and $\succ'_J \in \mathbf{P}_J$ such that

$$\begin{aligned} \varphi[\succ'_J, \succ_{-J}](i) &\succeq_i \varphi[\succ](i) \qquad \forall i \in J \text{ and} \\ \varphi[\succ'_J, \succ_{-J}](j) &\succ_j \varphi[\succ](j) \qquad \exists j \in J. \end{aligned}$$

Coalitional strategy-proofness is dominant strategy incentive compatibility for a group and is a cooperative property. Therefore, it has a verifiability problem when agents communicate their private information and action plan to each other in a coalition in the standard non-cooperative settings. On the other hand, in our domain, it also has a non-cooperative interpretation. It is equivalent to the combination of two non-cooperative axioms, non-bossiness and strategy-proofness (Pápai 2000). *Non-bossiness* (Satterthwaite and Sonnenschein 1981) of a mechanism requires that when an agent misreports his preferences and gets the same assignment that he was getting under truthful revelation, then he cannot change the allocation regarding the other agents, either. *Strategy-proofness* of a mechanism means that the truthful revelation of preferences is a weakly dominant strategy.

Formally, a mechanism φ is **non-bossy** if for all $\succ \in \mathbf{P}$, all $i \in I$, and all $\succ'_i \in \mathbf{P}_i$,

$$\varphi[\succ'_i, \succ_{-i}](i) = \varphi[\succ](i) \quad \Rightarrow \quad \varphi[\succ'_i, \succ_{-i}] = \varphi[\succ].$$

Formally, a mechanism φ is **strategy-proof** if for all $\succ \in \mathbf{P}$, there exist no $i \in I$ and $\succ'_i \in P$ such that

$$\varphi[\succ'_i, \succ_{-i}](i) \succ_i \varphi[\succ](i).$$

We state their relationship with the following lemma:

Lemma 1 *(Pápai 2000) A house-allocation mechanism is coalitionally strategy-proof if and only if it is strategy-proof and non-bossy.*

There is another property that is closely related to coalitional strategy-proofness. A mechanism is *(Maskin-)monotonic* if whenever the preferences change in such a way that the set of houses better than the assigned house weakly shrinks for each agent, then the matching assigned by the mechanism does not change (Dasgupta, Hammond and Maskin 1979). Formally, a mechanism φ is **Maskin-monotonic** if for all $\succ, \succ' \in \mathbf{P}$ and all $i \in I$,

$$\{h \in H : h \succeq_i \varphi[\succ](i)\} \supseteq \{h \in H : h \succeq'_i \varphi[\succ](i)\} \quad \Rightarrow \quad \varphi[\succ'] = \varphi[\succ].$$

We also say that for each such $\succ'$ and $\succ$, profile $\succ'$ is a **monotonic extension of $\succ$ under φ**.

The following lemma states the relationship of Maskin monotonicity and coalitional strategy-proofness in our domain.[5]

Lemma 2 *(Dasgupta, Hammond, and Maskin 1979) A house-allocation mechanism is Maskin-monotonic if and only if it is coalitionally strategy-proof.*

We will use these two equivalences in our proofs.

4 The Trading-Cycles-with-Brokers-and-Owners Algorithm

In this section, we introduce a new algorithm called *trading cycles with brokers and owners*. This algorithm is iterative and in each round of the algorithm, some agents are removed after being matched. The algorithm works as follows: In each round, it assigns the control rights of each unremoved house to some unremoved agent. This agent controls this house as an "owner" or as a "broker." In either case, this house cannot be matched in this round unless its control rights holder is matched. Since this agent can hold control rights of multiple houses and he is matched in the same round as one of his controlled houses, once he is removed, we need to designate an unremoved inheritor that will take control of his unremoved houses. Moreover, there are some occasions in which an agent can lose the control of a house from one round to the other although he is not removed.

[5] The result obtains because our domain consists of strong preferences and is "rich" in the sense of Dasgupta, Hammond, and Maskin (1979).

Thus, we need to define a proper control rights structure that is a function of the previously matched agents and their matches in order to define our algorithm formally. The assignment produced by our algorithm depends on this structure of control rights. Let us define this new concept first.

Definition: A **structure of control rights** (c, b) consists of

- a profile of **control functions** $c = (c_h : \mathcal{S}_{-h} \longrightarrow I)_{h\in H}$ such that for all h and all $\sigma \in \mathcal{S}_{-h}$, $c_h(\sigma) \in I - I_\sigma$, and
- a **brokered house function** $b : \mathcal{S}-\mathcal{M} \longrightarrow H\cup\{\emptyset\}$ such that for all $\sigma \in \mathcal{S}-\mathcal{M}$, if $|I_\sigma| = |I|-1$, then $b(\sigma) = \emptyset$.[6]

Fix a proper submatching σ. Let this submatching represent the agents removed previously and the houses that they were matched with. If $\sigma \in \mathcal{S}_{-h}$ for some house h, then h is not unmatched yet under σ. In this case, unmatched agent $c_h(\sigma)$ is said to **control** house $h \in H - H_\sigma$ at submatching σ. A house $h \in H - H_\sigma - \{b(\sigma)\}$ is called an **owned house**, or simply **o-house, at** σ; its controller $c_h(\sigma)$ is called the **owner of** h **at** σ; and the owner – o-house pair $(c_h(\sigma), h)$ is called an **o-pair at** σ.[7] If $b(\sigma) \neq \emptyset$, that is, there exists a brokered house at this submatching, then house $b(\sigma)$ is called the **brokered house**, or simply **b-house**, **at** σ; its controller $c_{b(\sigma)}(\sigma)$ is called the **broker at** σ; and the broker – b-house pair $\left(c_{b(\sigma)}(\sigma), b(\sigma)\right)$ is called the **b-pair at** σ.[8] On the other hand, if $b(\sigma) = \emptyset$, then there is no $\sigma-$b-house and all unmatched houses are $\sigma-$o-houses.

Observe that while there can be multiple o-pairs, there can be at most one b-pair, and moreover, each unmatched house is controlled by one and only one unmatched agent. While it is possible that some unmatched agents do not control any unmatched houses, some unmatched agent can control multiple unmatched houses.

For all control rights structures, the assignment of houses to agents is determined by an iterative algorithm that we refer to as the **trading-cycles-with-brokers-and-owners algorithm** (**TCBO algorithm** for short).

[6] In the sequel, we will develop compatibility conditions for the control rights structures.

[7] In this case, we will occasionally write that "h is a $\sigma-$o-house;" "$c_h(\sigma)$ is a $\sigma-$owner;" and "$(c_h(\sigma), h)$ is a $\sigma-$o-pair."

[8] In this case, we will occasionally write that "$b(\sigma)$ is the $\sigma-$b-house;" "$c_{b(\sigma)}(\sigma)$ is the $\sigma-$broker;" and "$\left(c_{b(\sigma)}(\sigma), b(\sigma)\right)$ is the $\sigma-$b-pair."

In the description of the algorithm, we will use some graph theoretical concepts. The directed graphs constructed in the algorithm have the unremoved agents and houses as the vertices; each house points to a single agent, and each agent points to a single house (thus determining the directed edges). A **cycle** of length n is a directed graph

$$h^1 \to i^1 \to h^2 \to ...h^n \to i^n \to h^1$$

such that $i^\ell \in I$ points to $h^{\ell+1} \in H$ and h^ℓ points to i^ℓ for $\ell \in \{1, ..., n\}$, and $i^\ell \neq i^{\ell'}$, $h^\ell \neq h^{\ell'}$ for all $\ell, \ell' \in \{1, ..., n\}$, $\ell \neq \ell'$. Whenever we talk about cycles of length n, we will identify superscripts using modulo n, thus $n+1$ and 1 are equivalent. In each such directed graph, there exists at least one cycle and no two cycles intersect.

We are ready to state a round of the TCBO algorithm:

> **Round $r = 1, 2, ...$ of the TCBO algorithm:** Let σ^{r-1} be the submatching of agents and houses removed before round r. Before the first round, the submatching of removed agents is empty, $\sigma^0 = \varnothing$.
>
> *Determination of intra-round trade graph:* Each unremoved house h points to the agent who controls it at σ^{r-1}. If there exists a σ^{r-1}-*broker*, he points to his first choice σ^{r-1}-*o-house.* Every other unremoved agent i points to his top choice house among the unremoved houses.
>
> *Removal of trading cycles:* There exists at least one cycle. We remove each agent in each cycle by assigning him the house he is pointing to.
>
> *Stopping rule:* We stop the algorithm if all agents are removed (matched). The resultant matching, σ^r, is then the outcome of the algorithm.

Since we assign at least one agent a house in every round, and since there are finitely many agents, the algorithm stops after finitely many rounds.

The terminology of owners and brokers is motivated by the trading analogy. In each round of the algorithm, an owner can either trade a house he controls for another house (in a cycle of several exchanges), or can leave in this round matched with a house he owns. A broker can trade the house he owns for another house (in a cycle of several exchanges), but cannot leave in this round matched with the house he brokers. One interpretation of this is that the owner can consume his house, but the broker cannot.

Our algorithm builds upon Gale's top-trading cycles idea (Shapley and Scarf 1974) but differs in one important aspect from all other variants of the top-trading cycles algorithm such as top-trading cycles algorithm with newcomers (Abdulkadiroğlu and Sönmez 1999), hierarchical exchange algorithm (Pápai 2000), top-trading cycles algorithm for school choice (Abdulkadiroğlu and Sönmez 2003) and top-trading cycles and chains algorithm (Roth, Sönmez, and Ünver 2004). All these algorithms may be interpreted as TCBO in which all control rights are ownership rights and there are no brokers. In particular, in all these algorithms, in every round, all remaining agents point to their remaining top choice house. Thus, they refer to the forming exchange cycles as *top-trading cycles*. In TCBO algorithm, all remaining owners point to their remaining top choice house, except the broker. He points to his remaining top choice *o-house* and he does not point to the *b-house* even if it is his top choice.

5 Examples

We start with an example of how the TCBO algorithm is executed:

Example 1 ***(Execution of the TCBO algorithm)*** *Let* $I = \{i_1, i_2, i_3\}$ *and* $H = \{h_1, h_2, h_3\}$. *Suppose the control rights structure is such that*

- h_1 *is owned by* i_1 *as long as* i_1, h_1 *are unmatched, is owned by* i_2 *when* i_2, h_1 *are unmatched and* i_1 *is matched, and is owned by* i_3 *when* i_3, h_1 *are unmatched and* i_1, i_2 *are matched,*
- h_2 *is owned by* i_2 *as long as* i_2, h_2 *are unmatched, is owned by* i_1 *when* i_1, h_2 *are unmatched and* i_2 *is matched, and is owned by* i_3 *when* i_3, h_2 *are unmatched and* i_1, i_2 *are matched,*
- h_3 *is controlled by* i_3*; he has the brokerage right as long as either* i_1 *and* i_2 *are unmatched and the ownership right when* i_1 *and* i_2 *are matched (notice that we do not need to specify who*

inherits h_3 *when* i_3 *is matched, because* i_3 *may be matched only in a cycle that also contains* h_3*).*

The above structure of control rights may be graphically represented as follows:

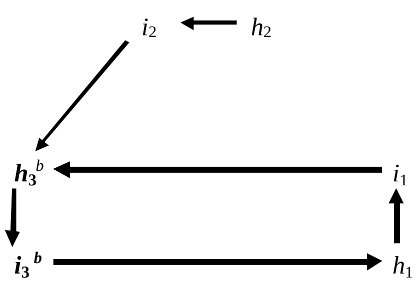
c_{h_1}	c_{h_2}	c_{h_3}
i_1	i_2	i_3 ^{b}
i_2	i_1	
i_3	i_3	

The ^{b} *sign, above, next to* i_3 *in* h_3*'s control right column, shows that* h_3 *is a b-house (when some agents other than* i_3 *who controls* h_3 *are unmatched). The preferences of the agents are given as follows:*

$$\text{agent 1: } h_3 \succ_{i_1} h_2 \succ_{i_1} h_1$$
$$\text{agent 2: } h_3 \succ_{i_2} h_2 \succ_{i_2} h_1$$
$$\text{agent 3: } h_3 \succ_{i_3} h_1 \succ_{i_3} h_2$$

We run the algorithm as follows:

Round 1. *O-house* h_1 *points to* $c_{h_1}(\varnothing) = i_1$*, o-house* h_2 *points to* $c_{h_2}(\varnothing) = i_2$*, b-house* $b(\varnothing) = h_3$ *points to* $c_{b(\varnothing)}(\varnothing) = i_3$*. Agents* i_1 *and* i_2 *point to* h_3 *and broker* i_3 *points to his first choice o-house, that is* h_1*. This directed graph is given below:*



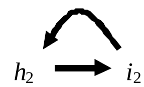
There exists one cycle

$$h_1 \to i_1 \to h_3 \to i_3 \to h_1,$$

and by removing it, we obtain

$$\sigma^1 = \{(i_1, h_3), (i_3, h_1)\}$$

Round 2. *O-house h_2 points to $c_{h_2}(\sigma^1) = i_2$ and agent i_2 points to h_2. This directed graph is given below:*



There exists one cycle $h_2 \to i_2 \to h_2$, and by removing it, we obtain

$$\sigma^2 = \{(i_1, h_3), (i_3, h_1), (i_2, h_2)\}.$$

This is a matching, since no agents are left.

We terminate the algorithm, the outcome of the mechanism is σ^2.

Remark 1 *The mechanism (i.e., the mapping from preference profiles to assignments) of Example 1 is equivalent to the mechanism of Example 6 in Pápai (2000), though Pápai's algorithm differs considerably from our trading-cycles-with-brokers-and-owners algorithm. Pápai shows that the mechanism is coalitionally strategy-proof and Pareto-efficient but does not satisfy her reallocation-proofness condition (cf. the introduction for the definition of reallocation-proofness). Thus, this example shows that our class of mechanisms is more general than Pápai's class.*

Example 2 ***(TCBO with Persistent Brokers)***

*A control rights structure (c, b) has **persistent brokers** (or be **strongly compatible**) if for all submatchings $\sigma \in \mathcal{S} - \mathcal{M}$, we have:*

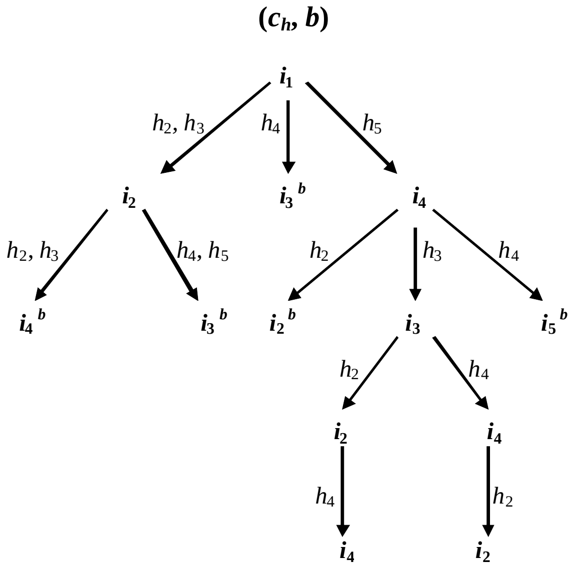
*C1. **(Persistence of ownership)** If i owns h at σ and if i and h are unmatched at $\sigma' \supset \sigma$, then i owns h at σ'.*

*C2. **(No ownership for brokers)** If k is a broker at σ, then k does not own any house at σ.*

*SC3. **(Persistence of brokerage)** If agent k brokers house f at σ and if agent k, house f, and an agent $i \neq k$ are unmatched at $\sigma' \supset \sigma$, then k brokers f at σ'.*

In Section 7, we will show that each TCBO with persistent brokers is coalitionally strategy-proof and Pareto-efficient.

Let us now look at what these strong compatibility conditions imply through the following problem. Suppose $I = \{i_1, i_2, ..., i_5\}$ and $H = \{h, h_2, ..., h_5\}$. Let (c, b) be a strongly compatible control rights structure. The structure regarding house h is represented through the following graph:



This graph can be interpreted as follows: House h is initially inherited by i_1, that is $c_h(\emptyset) = i_1$. If agent i_1 is assigned house h_2 or h_3 then the control right of h is inherited by agent i_2, when he is assigned h_4, the control right of h is inherited by i_3, and finally, when he is assigned h_5, the control right of h is inherited by agent i_4. That is,

$$c_h(\{(i_1, h_2)\}) = c_h(\{(i_1, h_3)\}) = i_2,$$
$$c_h(\{(i_1, h_4)\}) = i_3,$$
$$c_h(\{(i_1, h_5)\}) = i_4.$$

This tree structure shows the persistence of o-pairs and b-pairs (C1 and SC3). The ^{b} sign next to agent i_3 shows that house h becomes a b-house when it is inherited by i_3, moreover the broker status of i_3 persists until h_3 is removed. For strong compatibility (C2), we need i_3 be the last inheritor in subtree of c_h following submatching $\{(i_1, h_4)\}$ for all $h \in \{h_2, h_3, h_5\}$.

Similarly as we follow other submatchings, we obtain

$$c_h(\{(i_1, h_2), (i_2, h_3)\}) = c_h(\{(i_1, h_3), (i_2, h_2)\}) = i_4.$$
$$c_h(\{(i_1, h_2), (i_2, h_4)\}) = c_h(\{(i_1, h_3), (i_2, h_4)\})$$
$$= c_h(\{(i_1, h_2), (i_2, h_5)\}) = c_h(\{(i_1, h_3), (i_2, h_5)\}) = i_3.$$

In either case, h becomes a b-house, but its control rights are given to different agents, i_4 or i_3 respectively. We can interpret the remainder of the control rights structure of house h, similarly.

6 The Trading-Cycles-with-Brokers-and-Owners Mechanism

We are ready to formally define our mechanism class. We will define compatibility of control rights structures by maintaining Conditions C1-C2 of Example 2, and relaxing condition SC3.

Definition (Compatibility of control rights structures): A control rights structure (c, b) is **compatible** if for all submatchings $\sigma \in \mathcal{S} - \mathcal{M}$,

C1. **(Persistence of ownership)** If agent i owns house h at σ, and i and h are unmatched at $\sigma' \supset \sigma$, then i owns h at σ'.

C2. **(No ownership for brokers)** If agent k is a broker at σ, then k does not own any house at σ.

C3. **(Limited persistence of brokerage)** If agent k brokers house f at σ, agent $j \neq k$ and house $g \neq f$ are unmatched at σ, and k does not broker f at submatching $\sigma \cup \{(j, g)\}$, then either

- (**Broker-to-heir transition**) (i) there is exactly one agent i who owns a house both at σ and $\sigma \cup \{(j, g)\}$, (ii) agent i owns house f at $\sigma \cup \{(j, g)\}$, and (iii) at submatching $\sigma \cup \{(j, g), (i, f)\}$, agent k owns all houses that i owns at σ,

  or

- (**Direct exit from brokerage**) there is no agent who owns a house at both σ and $\sigma \cup \{(j, g)\}$.

Each *compatible* pair (c, b) induces a **trading-cycles-with-brokers-and-owners mechanism** (**TCBO mechanism** for short). We denote it as $\psi^{c,b}$. Its outcome is found through the TCBO algorithm that was introduced earlier. All control rights structures discussed in Examples 1-3 are compatible, hence the induced mechanisms are TCBO.

Under the compatibility conditions, a b-pair may not persist as larger submatchings are fixed. This is the only difference from the strong compatibility conditions introduced in Example 2 for persistent brokers.

When exactly one previous owner i remains unmatched at the larger submatching with one more agent matched, a broker k can *transit* from being a *broker* to being a *heir* to this owner. After the broker k loses his brokerage privilege, the ex-b-house is owned by the owner i. However, we need a protection for the ex-broker k in this case, since he could have used his brokerage privilege in a trade earlier to get a house that i owned. So, if i gets the ex-b-house and leaves, then the inheritance system defined in C3 guarantees that the ex-broker k owns the owner i's houses that he owned at the instant the broker k went into transition. Thus, we refer to the ex-broker as an *heir to* i in this case.

A broker can *directly exit* from *brokerage* without any protection at a larger submatching only when there is a single owner and the larger submatching is obtained by this owner being matched. In fact, *broker-to-heir transition* and *direct exit from brokerage* are not different conditions. They are just different interpretations of the same situation. Under the TCBO algorithm, at a submatching σ, suppose that there is a single owner i unmatched and some unmatched agent k brokers some unmatched house f. Observe that we can replicate this situation without any broker, i.e., i owning all unmatched houses at σ and k owning all unmatched houses when i is matched with the so-called "b-house" f. Thus, *being a broker at* σ *and then directly exiting brokerage at* $\sigma \cup \{(i, g)\}$ (for any unmatched house g at σ) for agent k is inherently equivalent to an immediate *broker-to-heir transition at* σ.

On the other hand, in all other cases, the b-pair persists as larger submatchings are obtained, and this is the *limited persistence of brokerage* defined in C3. The following example illustrates the limited persistence of b-pairs.

Example 3 ***Limited persistence of b-pairs.*** *Consider an environment with four agents:* $I = \{j, i_1, i_2, k\}$ *and four houses:* $H = \{h_1, h_1', h_2, f\}$*, and a TCBO mechanism* $\psi^{c,b}$ *whose control rights structure* (c, b) *is illustrated by the following table explained below (cf. also Example 1 for the use of tables):*

c_{h_1}	$c_{h_1'}$	c_{h_2}	c_f
i_1	i_1	i_2	$k\ ^b$
j	j	i_1	$(i_1, h_1):\quad i_2\ ^{broker \to heir}$
i_2	i_2	j	j
k	k	k	k

In particular,

Houses h_1, h_1' are owned by agent i_1, when he is matched the unmatched of the two houses is owned by j, then i_2, and k. House h_2 is owned by i_2, then i_1, j, and k.

Agent k has the brokerage right over f initially (i.e., at the empty submatching). He remains the broker as long as he is unmatched with one exception. The broker-to-heir transition occurs after the assignment (i_1, h_1). At this point i_2 is the only remaining owner left from the previous round (hence C3 is satisfied). Then, agent i_2 becomes the owner of f. The superscript, broker→heir, next to i_2 denotes this transition in the control rights structure of house f.

Notice, that C2 is satisfied as broker k is the last inheritor of all o-houses.

Remark 2 *The above compatible control rights structure (c, b) cannot be interpreted as a strongly compatible structure with persistent brokers (introduced in Example 2):*

Proof of Remark 2. By way of contradiction, let us assume that there is a TCBO mechanism with persistent brokers ϕ that produces the same allocation as $\psi^{c,b}$ for each profile of agents' preferences.

Let us define the following classes of preference profiles. For all submatchings $\sigma \in \mathcal{S} - \mathcal{M}$ and all houses $h \in H - H_\sigma$, let $\mathbf{P}[\sigma, h] \subset \mathbf{P}$ be the set of preference profiles such that for all $\succ \in \mathbf{P}[\sigma, h]$,

- for all $i \in I_\sigma$,

  $$\sigma(i) \succ_i g \text{ for all } g \in H - \{\sigma(i)\},$$

  that is, each i matched at σ ranks house $\sigma(i)$ as his first choice, and

- for all $i \in I - I_\sigma$,

  $$h \succ_i g \succ_i g' \text{ for all } g \in H - H_\sigma - \{h\} \text{ and all } g' \in H_\sigma,$$

  that is, each i unmatched at σ ranks (i) house h as his first choice and (ii) the houses matched at σ lower than the houses unmatched at σ.

For all submatchings $\sigma \in \mathcal{S}$ and all houses $h, h' \in H - H_\sigma$, let $\mathbf{P}[\sigma, h, h'] \subset \mathbf{P}$ be the set of preference profiles such that for all $\succ \in \mathbf{P}[\sigma, h, h']$,

- for all $i \in I_\sigma$,

  $$\sigma(i) \succ_i g \text{ for all } g \in H - \{\sigma(i)\},$$

  that is, each i matched at σ ranks house $\sigma(i)$ as his first choice, and

- for all $i \in I - I_\sigma$,

$$h \succ_i h' \succ_i g \succ_i g' \text{ for all } g \in H - H_\sigma - \{h, h'\} \text{ and all } g' \in H_\sigma,$$

that is, each i unmatched at σ ranks (i) house h as his first choice, (ii) house h' as his second choice, and (iii) the houses matched at σ lower than the houses unmatched at σ.

First, notice that at the empty submatching, k is the *broker* of f in the TCBO mechanism with persistent brokers ϕ. It is so, because f is not owned by any agent at the empty submatching $\varnothing$ as $(\phi[\succ])^{-1}(f) = \left(\psi^{c,b}[\succ]\right)^{-1}(f)$ varies with $\succ \in \mathbf{P}[\varnothing, f]$. Hence, there is an agent who has the brokerage right over f and it must be k as $\phi[\succ](k) = \psi^{c,b}[\succ](k) = g$ for all $\succ \in \mathbf{P}[\varnothing, f, g]$ and all $g \in \{h_1, h_1', h_2\}$.

Second, consider the submatching $\sigma = \{(i_1, h_1)\}$ and a preference profile $\succ \in \mathbf{P}[\sigma, f, h_1']$. In mechanism ϕ, agent k would continue to be the broker of f at σ and thus

$$\phi[\succ](k) = h_1'.$$

However,

$$\psi^{c,b}[\succ](k) = h_2,$$

and thus, $\phi \neq \psi^{c,b}$ for all TCBO mechanisms with persistent brokers ϕ. **QED**

Remark 3 *When $|I| = 3$, brokers are persistent in compatible control rights structures.*

Proof of Remark 3. First notice that with only two agents left unmatched, having one owner and one broker is equivalent to having one owner and one agent with no rights. That is, even if the broker controls the b-house, the owner will definitely get it if he wants it. Thus, this is observationally equivalent to the owner also owning the "b-house".

Next, suppose there is a broker and the other two agents are owners. Even if the broker loses his status after one of the owners get matched, by the reasoning in the previous paragraph, it will be equivalent to broker remaining as a broker. Thus, in this case, *strong compatibility conditions* are equivalent to *compatibility conditions*. Since with three agents, this is the only case when a broker can lose his status under the compatible control rights structures, the brokers are persistent. **QED**

The above remark and Theorems 1-4, which are proved next, imply that

Corollary 1 *If $|I| = 3$, then a mechanism is coalitionally strategy-proof and Pareto-efficient if and only if it is TCBO with persistent brokers.*

In the next section, we will show that a mechanism is coalitionally strategy-proof and Pareto-efficient if and only if it is equivalent to a TCBO mechanism.

7 The Main Result

In this section, we first show that trading-cycles-with-brokers-and-owners mechanism satisfy coalitional strategy-proofness and Pareto-efficiency (Theorems 1-3). Theorem 4 shows that any coalitionally strategy-proof and Pareto-efficient direct mechanism is TCBO. The proof of this result is constructive, given any coalitionally strategy-proof and Pareto-efficient mechanism φ it shows how to construct the corresponding control rights structure (c, b), and verifies the resultant TCBO $\psi^{c,b} = \varphi$.

Theorem 1 *The trading-cycles-with-brokers-and-owners algorithm produces a Pareto-efficient matching for all control rights structures that satisfy C2.*

Proof of Theorem 1. Consider the execution of the TCBO algorithm. Observe that each agent removed in Round 1 gets his top choice house, possibly except a broker removed. If there exists a broker, the b-house is his top choice, and he is removed in Round 1, then the broker receives his second choice. Observe that by C2, a broker can only be removed together with a b-house, since he does not own any house. Thus, the b-house is assigned to an agent who had the b-house as his top choice, hence if the broker were to be assigned the b-house, he would make this other agent worse off. Thus, nobody removed in Round 1 can be made better off without making somebody removed in Round 1 worse off. Observe that each agent removed in Round 2 gets his remaining top choice, possibly except a broker. If there exists a broker, the b-house is his remaining top choice, and he is removed in Round 2, then he receives his second remaining choice. In this case, the b-house is also assigned to an agent in Round 2 (by C2), and this agent prefers it more than any house assigned in the second round. Thus, nobody removed in Round 2 can be made better off without making somebody removed in Round 1 or 2 worse off. We continue iteratively, showing that the outcome of the TCBO algorithm is Pareto-efficient. **QED**

Next, we make three observations about the TCBO algorithm:

Observation 1. At all rounds r of the algorithm, the control rights structure (i.e., current set of o-pairs, control rights owners, and the possible b-pair) depends only on the removed submatching σ^r. In other words, if two preference profiles $\succ$ and $\succ'$ induce the same σ^r at round r, then they induce the same control rights structure.

Pápai (2000) uses the name "twin inheritance rule" to refer to an analogous observation about her subclass of the class of mechanisms studied in this paper.

Observation 2. If an agent i is unmatched at a round r of the algorithm under preference profiles $[\succ_i, \succ_{-i}]$ and $[\succ_i', \succ_{-i}]$, then the same submatching is left before r, that is $\sigma^{r-1}[\succ_i, \succ_{-i}] = \sigma^{r-1}[\succ_i', \succ_{-i}]$.

Observation 3. If an agent i is unmatched at a round r of the algorithm under preference profiles $[\succ_i, \succ_{-i}]$ and $[\succ_i', \succ_{-i}]$, then the control rights structure is the same under $[\succ_i, \succ_{-i}]$ and $[\succ_i', \succ_{-i}]$.

In particular, an agent cannot affect, by submitting different preferences, when he becomes a broker, enters broker-to-heir transition, and becomes an owner.

Next, we will prove that TCBO mechanisms (that is, the mappings that find the outcome of each problem through a compatible control rights structure using the TCBO algorithm) are also strategy-proof and non-bossy, implying through Lemma 1 that they are coalitionally strategy-proof.

Theorem 2 *Every trading-cycles-with-brokers-and-owners mechanism is* strategy-proof.

Proof of Theorem 2. Let $\psi^{c,b}$ be a TCBO mechanism. Let $\succ$ be a preference profile. We fix an agent $i \in I$. We will show that i cannot benefit by submitting $\succ_i' \neq \succ_i$ while the other agents submit $\succ_{-i}$. Let s be the round i leaves (with house h) at $\succ_i$ and s' be the time i leaves (with h') at $\succ_i'$ in the algorithm. We will consider two cases.

Case 1. $s \leq s'$**:** At round s, same houses and agents are in the market at both $\succ_i$ and $\succ_i'$ by Observation 3. If i is not a broker at time s under $\succ_i$, then, by submitting $\succ_i$, agent i gets the top house among the remaining ones in round s, implying that he cannot be better off by submitting $\succ_i'$.

Assume now that i is a broker at time s under $\succ_i$. Let f be the b-house at time s. If f is not agent i's top choice house remaining under $\succ_i$, then by submitting $\succ_i$, agent i gets the top house among the remaining ones in round s, implying that he cannot be better off by submitting $\succ_i'$.

It remains to consider the situation in which f is broker i's top choice remaining house, and to show that i cannot get f by submitting the profile $\succ_i'$. For an argument through contradiction, assume that under $\succ_i'$ agent i leaves at round s' with house f.

Notice that under $\succ_i$, there is an agent j who is matched with house f at time s, because i is a broker and he leaves in the same cycle in round s. At this time, j is an owner of some o-house h_j, and f is his top choice house. By Observation 3, the control rights structure at round s is the same under both $\succ_i$ and $\succ_i'$. Hence, i is also a broker at time s after submitting $\succ_i'$, and j is an owner of h_j. Moreover, j's top choice is still house f. That means that under $\succ_i'$ agent j will stay unmatched till $s'+1$. Since agent i leaves with f at s', he cannot be the broker of f at this round, because a broker cannot leave with the b-house, while another owner j is unmatched. Thus, there is a round $s'' \in \{s+1, ..., s'\}$ at which agent i stops being the broker of f. Since f is still unmatched at this round, there is a broker-to-heir transition between $s''-1$ and s'' (by C3). Because j is an owner of h_j at both $s''-1$ and s'', he would have inherited f at s'' (by C3). Then, however, j would have left with f at s'', as f is j's top choice among houses left at s (and hence those left at s''). A contradiction.

Case 2. $s > s'$**:** At round s', same houses and agents are in the market at both $\succ_i$ and $\succ_i'$ by Observation 3.

Consider round s' at both $\succ_i$ and $\succ_i'$. Under $\succ_i'$, agent i points to house $h' = h^1$ that points to agent i^1 that points to ... that points to object h^n that points to agent $i = i^n$ (and this cycle leaves at round s'). If the cycle is trivial ($n = 1$) and h' points back to i, then (i, h') is an o-pair. Since o-pairs persist by C1, it will be an o-pair at $s > s'$, and thus at round s, agent i would leave with a house at least as good as h'.

In the sequel, assume that there is at least one other agent i^n in the cycle (that is $n \geq 2$).

If all pairs $\left(i^\ell, h^\ell\right)$ are o-pairs, for all $\ell \in \{1, ..., n\}$, then the chain $h' = h^1 \to i^1 \to h^2 \to ... \to h^n \to i$ will stay in the system as long as i is in the system (by persistency of o-pairs implied through C1). Thus, at round s agent i would leave with a house at least as good as h' under $\succ_i$.

If $\left(i^{\ell}, h^{\ell}\right)$ is a b-pair for some $\ell \in \{1, ..., n\}$, then the chain $h' = h^1 \to i^1 \to h^2 \to ... \to h^n \to i$ will stay in the system as long as $\left(i^{\ell}, h^{\ell}\right)$ continues to be a b-pair (since there are no other b-pairs and the o-pairs persist by C1). If it is a b-pair at round s under $\succ_i$, then we are done, since the same cycle would have formed. Thus suppose that at a round $s'' \in \{s'+1, ..., s\}$ broker i^{ℓ} loses his broker status. Because $n \geq 2$, agent $i^{\ell+1}$ is an owner both at rounds $s''-1$ and s''. Hence, the loss of brokerage status means that i^{ℓ} enters broker-to-heir transition. We must then have $n = 2$ (since by C3, only 1 previous owner can remain unmatched during broker-to-heir transition). There are two cases: either i^1 owns $h^1 = h'$ and h^2 (and $i^2 = i^{\ell}$ is the heir) or $i^2 = i$ owns h^1 and h^2. In the former case, i^1 who wants h^2, will leave with it at round s'' under $\succ_i$, and i will inherit $h^1 = h'$ at $s''+1$ by the definition of an inheritance system and by C3. In the latter case, i owns $h^1 = h'$ already at round s''. In both cases, at $s \geq s''$ agent i can only leave with a house at least as good as h' under $\succ_i$. **QED**

Theorem 3 *Every trading-cycles-with-brokers-and-owners mechanism is* non-bossy*.*

The proof of this theorem is in Appendix A. In lieu of a heuristic argument, we state and prove the following weaker proposition in which the assumption C3 is replaced by a stronger assumption SC3 (formulated in Example 2).

Proposition 1 *Every TCBO mechanism with persistent traders (i.e., with control rights structures satisfying C1, C2, and SC3) is* non-bossy*.*

Proof of Proposition 1. Let $\psi^{c,b}$ be a TCBO mechanism with persistent brokers (that is, (c, b) is strongly compatible and satisfies C1, C2 and SC3). Let $\succ$ be a preference profile. We fix an agent $i_* \in I$. We will show that i_* cannot change the allocation of the other agents by submitting $\succ'_{i_*} \neq \succ_{i_*}$ and obtaining the same house

$$\psi^{c,b}\left[\succ'_{i_*}, \succ_{-i_*}\right](i_*) = \psi^{c,b}\left[\succ\right](i_*) = h_*.$$

Let $\succ' = \left[\succ'_{i_*}, \succ_{-i_*}\right]$. Let s be the round i_* leaves (with house h_*) under $\succ$ and s' be the round i_* leaves (with h_*) under $\succ'$ in the TCBO algorithm. Without loss of generality, suppose $s' \geq s$. By Observation 3, we have $\sigma^r[\succ'] = \sigma^r[\succ]$ for all $r \in \{1, 2, ..., s-1\}$.

We will show that each cycle removed in a round $r \geq s$ of the algorithm under $\succ$ will be removed in some round r' of the algorithm under $\succ'$. The argument will be by an induction with respect to r.

For $r = s$, the same houses and agents are in the market under both $\succ$ and $\succ'$. By Observation 3, each cycle that is formed in round s under $\succ$, will form under $\succ'$ in round s, as well, except possibly the cycle in which i_* leaves in round s under $\succ$. Let us denote this cycle by $C_* = h_* \rightarrow i_*^1 \rightarrow h_*^2 \rightarrow \ldots h_*^n \rightarrow i_* \rightarrow h_*$. We show that cycle C_* forms under $\succ'$ in some round r'. By C1 and SC3, the same chain $h_* \rightarrow i_*^1 \rightarrow h_*^1 \rightarrow \ldots i_*^{n-1} \rightarrow h_*^n \rightarrow i_*$ will remain in the algorithm, as long as i_* is unmatched. Moreover, since i_* gets h_* under $\succ'$, he will point to h_* in round $s' > s$ and the same cycle C_* will form and will be cleared.

For the inductive step, fix round $r > s$ and assume that all cycles cleared in all rounds before r under $\succ$ are also cleared in some round of the algorithm under $\succ'$. Let

$$C = h^1 \rightarrow i^1 \rightarrow h^2 \rightarrow \ldots h^n \rightarrow i^n \rightarrow h^1$$

be a cycle that formed in round r under $\succ$. Let r' be the earliest round that one of the agents in C was removed under $\succ'$, and let i^ℓ be an agent removed at r' under $\succ'$. By the inductive step, i^ℓ gets house $h^{\ell+1}$ or worse under $\succ'$, and thus $h^{\ell+1}$ is removed at $r'' \leq r'$.[9] Let $\nu = \sigma^{r-1}[\succ] \cup \sigma^{r''-1}[\succ']$. House $h^{\ell+1}$ is controlled by $i^{\ell+1}$ at ν by C1 and SC3. Let $j^{\ell+1}$ be the agent that controls house $h^{\ell+1}$ at $\sigma^{r''-1}[\succ']$. By C1 and SC3, we either have $i^{\ell+1} = j^{\ell+1}$ or $j^{\ell+1}$ is matched at $\sigma^{r-1}[\succ]$. The latter case cannot happen. Indeed, $j^{\ell+1}$ is matched at $\sigma^{r''}[\succ]$ in a cycle containing $h^{\ell+1}$. If $j^{\ell+1}$ was matched at $\sigma^{r-1}[\succ]$ then the inductive assumption stating that all cycles cleared before round r under $\succ$ are also cleared under $\succ'$, ha contradiction. Hence, $i^{\ell+1} = j^{\ell+1}$ owns $h^{\ell+1}$ at $\sigma^{r''-1}[\succ']$, and $i^{\ell+1}$ is removed at r'' under $\succ'$. Thus, $r'' = r'$. By repeating the argument we conclude that $i^{\ell+2}$ owns $h^{\ell+2}$ at $\sigma^{r'-1}[\succ']$ and is removed at r' under $\succ'$, etc. Thus, agents i^ℓ, for all $\ell \in \{1, ..., n\}$, are removed at r', while controlling h^ℓ and desiring $h^{\ell+1}$. Thus, the cycle C is removed at r' under $\succ'$. **QED**

Finally, we state the converse of Theorems 1 – 3, completing this section:

Theorem 4 ***(Implementation Result)*** *If a mechanism is coalitionally strategy-proof and Pareto-efficient then it is a trading-cycles-with-brokers-and-owners mechanism.*

[9] The superscripts are modulo n, that is $n + 1 = 1$.

The proof of this theorem is in Appendix B. In the proof we fix a coalitionally strategy-proof and Pareto-efficient direct mechanism φ and construct a TCBO mechanism $\psi^{c,b}$ that is equivalent to φ.

We first construct the candidate control rights structure (c, b) using the set of profiles $\mathbf{P}^*[\sigma, h] = \cup_{h' \in (H - H_\sigma) - \{h\}} \mathbf{P}[\sigma, h, h']$ where $\mathbf{P}[\sigma, h, h']$ was introduced in the proof of Remark 2. It is straightforward to see that in a TCBO mechanism ψ, a σ-o-house h owned by i satisfies the property

$$\psi[\succ](i) = h \text{ for all } \succ \in \mathbf{P}^*[\sigma, h].$$

We use this property to define candidate o-houses and owners. A house $h \in H$ is a candidate o-house at $\sigma \in \mathcal{S}$ if it is unmatched at σ and the agent $\varphi[\succ]^{-1}(h)$ is constant across all $\succ \in \mathbf{P}^*[\sigma, h]$; then the agent $\varphi[\succ]^{-1}(h)$ is the candidate owner of h at σ. Furthermore, a σ-b-house f satisfies the property

$$\varphi[\succ]^{-1}(f) \neq \varphi[\succ']^{-1}(f) \text{ for some } \succ, \succ' \in \mathbf{P}^*[\sigma, f],$$

and broker k satisfies

$$\varphi[\succ](k) = h \text{ for all } \succ \in \mathbf{P}[\sigma, f, h].$$

We use this properties to define candidate b-houses and brokers. A house $f \in H$ is a candidate b-house at $\sigma \in \mathcal{S}$ if it is unmatched at σ and there exist some $\succ, \succ' \in \mathbf{P}^*[\sigma, f]$, such that $\varphi[\succ]^{-1}(f) \neq \varphi[\succ']^{-1}(f)$. An agent k is a candidate broker of house f at σ, if f is a candidate b-house at σ and for all $\succ \in \mathbf{P}^*[\sigma, h]$, house $\varphi[\succ](k)$ is the second choice of k in $\succ_k$.

We then show that the resultant control rights structure (c, b) is well defined and compatible. This is the hardest part of the proof. Finally, we show that the induced TCBO mechanism $\psi^{c,b}$ implements φ, that is, $\varphi[\succ] = \psi^{c,b}[\succ]$ for each preference profile $\succ$.

8 Market Design, Private Endowments, and Individual Rationality

In this section, we will introduce some market design problems which have relevance in both theory and application. These are modifications of the house allocation problem. In each such problem, there are also agents with private endowments. We will give characterizations in this domain using our main result.

Let $I^E \subseteq I$ be the set of agents each of whom **occupies** at least one house. We refer to set I^E as the set of **existing tenants**. Let the houses $h_{i,1}, ..., h_{i,\ell(i)}$ for some $\ell(i) \geq 1$ be occupied by agent

$i \in I^E$. Let $H^O = \{h_{i,1}, ..., h_{i,\ell(i)}\}_{i \in I^E} \subseteq H$ be the set of **occupied houses**. Preferences etc. are defined as before. Let $\succ$ be a preference profile. **A house allocation and exchange problem** is a list $\left\langle H, I, \left(i, h_{i,1}, ..., h_{i,\ell(i)}\right)_{i \in I^E}, \succ \right\rangle$. Matchings and mechanisms are defined in the same manner as before. We fix H, I and $\left(i, h_{i,1}, ..., h_{i,\ell(i)}\right)_{i \in I^E}$ so that a problem is defined just through its preference profile $\succ$. This problem has an ownership structure in its fundamentals unlike the house allocation problem.

Abdulkadiroğlu and Sönmez (1999) introduce the *house allocation problem with existing tenants* which is the special instance of our problem when each existing tenant occupies a single house. The house allocation problem with existing tenants is modeled after the real-life dormitory allocation problems in the US college campuses. In each such college, at the beginning of the academic year, there are new senior, junior, sophomore students, each of whom already occupies a room from the last academic year. There are vacated rooms by the graduating class and there are new freshmen who would like to obtain a room, though they do not currently occupy any. Another application of this model is kidney exchange with strict preferences (Roth, Sönmez, and Ünver 2004).

Besides Pareto efficiency and coalitional strategy-proofness, another important property of mechanisms of this problem domain is individual rationality. A matching is **individually rational**, if it assigns each existing tenant a house that is at least as good as his all occupied houses. Formally a matching μ is individually rational if

$$\mu(i) \succeq_i h_{i,\ell} \qquad \forall i \in I^E \text{ and } \ell \in \{1, ..., \ell(i)\}.$$

A mechanism is **individually rational** if it always selects an individually rational matching. If a mechanism is not individually rational, an existing tenant may opt out from trading and the mechanism may result with an inefficient outcome.

First, we state the following remark about the individual rationality of TCBO mechanisms:

Remark 4 *In house allocation and exchange problems, a TCBO mechanism is* individually rational *if and only if each exiting tenant is given the initial ownership rights of all of his occupied houses.*

Proof of Remark 4. Let ψ be an individually rational TCBO mechanism. Let $i \in I^E$ and $\succ \in \mathbf{P}[\varnothing, h_{i,\ell}]$ for some $\ell \in \{1, ..., \ell(i)\}$. Thus, $\psi[\succ](i) = h_{i,\ell}$ by individual rationality of ψ. Two cases are possible:

If $h_{i,\ell}$ is a $\varnothing$-o-house under ψ, then clearly agent i $\varnothing$-owns $h_{i,\ell}$.

If $h_{i,\ell}$ is a $\varnothing$-b-house under ψ, then agent i should be the $\varnothing$-owner of all other houses so that he can get his occupied house $h_{i,\ell}$ at each such profile $\succ \in \mathbf{P}[\varnothing, h_{i,\ell}]$. But then $I^E = \{i\}$, because i would have gotten $h_{j,k}$ at all preference profiles in $\mathbf{P}[\varnothing, h_{j,k}]$ for all of the other existing tenants $j \neq i$ and their occupied houses $k \in \{1, ..., \ell(j)\}$, contradicting ψ is individually rational . With $I^E = \{i\}$, this TCBO mechanism is equivalent to the one at which agent i $\varnothing$-owns all houses.

Conversely, suppose ψ is a TCBO mechanism at which at each existing agent i $\varnothing$-owns all of his occupied houses. Thus for each $i \in I^E$, $\psi[\succ](i) \succeq_i h_{i,\ell}$ for all $\ell \in \{1, ..., \ell(i)\}$, for all preference profiles $\succ$, showing that ψ is individually rational. **QED**

Next, we state our characterization result in this domain:

Proposition 2 *In house allocation and exchange problems, a mechanism is* individually rational*,* Pareto-efficient*, and* coalitionally strategy-proof *if and only if it is a TCBO mechanism that assigns each existing tenant the initial ownership rights of all of his occupied houses.*

Proof of Proposition 2. Let φ be an individually rational, Pareto-efficient and coalitionally strategy-proof mechanism for the allocation and exchange with problem. By Theorem 4 there exists a compatible (c, b) pair such that $\varphi = \psi^{c,b}$. By Remark 4, $\psi^{c,b}$ should be equivalent to a TCBO mechanism that assigns each existing tenant the initial ownership rights of all of his occupied houses.

Converse of it follows from Theorems 1-3. **QED**

We obtain another powerful and non-trivial characterization for another subdomain of problems:

Proposition 3 *In house allocation and exchange problems where each agent is an existing tenant, a mechanism is* individually rational*,* Pareto-efficient*, and* coalitionally strategy-proof *if and only if it is a TTC mechanism with hierarchical exchange that assigns all agents the initial ownership rights of their occupied houses.*

Proof of Proposition 3. Let φ be an individually rational, Pareto-efficient and coalitionally strategy-proof mechanism for the house allocation and exchange problem where each agent is an existing tenant. By Proposition 2**,** there exists an individually rational and compatible (c, b) pair such that $\varphi = \psi^{c,b}$. Since each patient owns* at least one donor under φ, by Lemma 11 (see Appendix B), there exists no broker* for φ.[10] By equivalence of φ to $\psi^{c,b}$, brokers and brokers* are equivalent

[10] See Appendix B for the formal definition of owner* and broker*.

at each $\sigma \in \mathcal{S} - \mathcal{M}$. Thus, there is no broker for $\psi^{c,b}$ implying that $\psi^{c,b}$ is a TTC mechanism with hierarchical exchange that assigns all agents the initial ownership rights of their occupied houses.

Converse of it follows from Theorems 1-3 and Proposition 2. **QED**

This result is a generalization of the result stated by Ma (1994). A *housing market* (Shapley and Scarf, 1974) is a house allocation and exchange problem that has only existing tenants, who each occupies a single house, and has no vacant houses. Ma characterizes the TTC mechanism that assigns the agents initial ownership rights of their occupied houses as the only mechanism that is individually rational, strategy-proof, and Pareto-efficient.

Another application of this restricted domain stated in the hypothesis of Proposition 3 is the *kidney exchange problem with good Samaritan donors* (Sönmez and Ünver, 2006): Kidney transplant patients are the agents and live kidney donors are the houses. All agents are existing tenants. Each agent has a single directed live donor (i.e., single occupied house), who would like to donate a kidney if his paired-donor receives a transplant in return. The unattached donors are good Samaritan donors (i.e., vacant houses) who would like to donate a kidney to any patient. In the US, good Samaritan donors have been the driving force behind kidney exchange since 2006. Many regional programs such as Alliance for Paired Donation (centered in Toledo, Ohio) and New England Program for Kidney Exchange (centered in Newton, Massachusetts) have used good Samaritan donors in majority of kidney exchanges that they conducted since 2006 (cf. Rees et al. 2009).

A note will be useful about coalitional strategy-proofness in this context. The doctors of the patients are the ones who have the information about patients' preferences over kidneys and it is well known that doctors (or transplant centers) themselves manipulate the system, if it will benefit their patients.[11] Hence, if kidney exchange mechanisms are coalitionally strategy-proof, a doctor (or a transplant center) will not be able to manipulate the mechanism on behalf of his (or its) patients without hurting at least one of them.

[11]For example, deceased-donor queue procedures are frequently gamed, by physicians acting as advocates for their patients. In particular, on July 29, 2003 two Chicago hospitals settled a Federal lawsuit alleging that some patients had been fraudulently certified as sicker than they were to move them up on the liver transplant queue (Warmbir, 2003).

A Appendix: Proof of Theorem 3

Proof of Theorem 3. Let $\psi^{c,b}$ be a TCBO mechanism. Fix an agent $i_* \in I$ and two preference profiles $\succ = [\succ_{i_*}, \succ_{-i_*}]$ and $\succ' = [\succ'_{i_*}, \succ_{-i_*}]$ such that

$$h_* = \psi^{c,b}[\succ'](i_*) = \psi^{c,b}[\succ](i_*).$$

Let s be the round i_* leaves (with house h_*) submitting $\succ_{i_*}$ and s' be the time i_* leaves (with h_*) submitting $\succ'_{i_*}$. By symmetry, it is enough to consider the case $s \leq s'$. In order to show that

$$\psi^{c,b}[\succ](i) = \psi^{c,b}[\succ'](i) \qquad \forall\, i \in I,$$

we will prove the following stronger statement:

Hypothesis: If a cycle of agents $h^1 \to i^1 \to h^2 \to ... \to h^n \to i^n \to h^1$ forms and is removed at round r when preferences $\succ$ were submitted, then either

- same cycle $h^1 \to i^1 \to h^2 \to ... \to h^n \to i^n \to h^1$ forms when preferences $\succ'$ are submitted, or
- $n = 2$ and two cycles $h^2 \to i^1 \to h^2$ and $h^1 \to i^2 \to h^1$ form when preferences $\succ'$ are submitted, or
- $n = 1$ and there exists agent $j \neq i^1$ and house $h \neq h^1$ such that the cycle $h \to i^1 \to h^1 \to j \to h$ forms when preferences $\succ'$ are submitted.

Whenever in the proof we encounter cycles of length n, the superscripts on houses and agents will be understood to be modulo n, that is $i^{n+1} = i^1$ and $h^{n+1} = h^1$.

By Observation 3, the above hypothesis is true for all $r < s$. The proof for $r \geq s$ will proceed by induction over the round r.

Initial step. Consider $r = s$. Under $\succ$, house h^1_* points to agent $i_* = i^1_*$ points to house $h_* = h^2_*$ that points to agent i^2_* that points to ... that agent i^n_* that points to house h^1_*, and the cycle

$$h^1_* \to i^1_* \to h^2_* \to ... \to h^n_* \to i^n_* \to h^1_*$$

is removed at round s. Observation 3 implies that the same houses and agents are in the market at time s under both $\succ$ and $\succ'$ and that all agents from $I_{\sigma^s[\succ]} - \{i^1_*, ..., i^n_*\}$ are matched by $\sigma^s[\succ']$ in the same way as in $\sigma^s[\succ]$.

Observation 3 also implies that the chain $h_*^2 \to ... \to h_*^n \to i_*^n \to h_*^1 \to i_*^1$ forms at round s under preferences $\succ'$.

If all pairs (i_*^ℓ, h_*^ℓ), for all $\ell \in \{2, ..., n\}$, are $\sigma^s[\succ]$-o-pairs, then they are $\sigma^s[\succ']$-o-pairs and the chain $h_*^2 \to ... \to h_*^n \to i_*^n \to h_*^1 \to i_*^1$ will stay in the system as long as i_*^1 is in the system (by persistency of o-pairs through C1). Thus, at s' all agents $i_*^1, ..., i_*^n$ would leave with same houses as under $\succ$.

If $n > 1$, and (i_*^ℓ, h_*^ℓ) is a b-pair for some $\ell \in \{2, ..., n\}$, then the chain $h_*^2 \to ... \to h_*^n \to i_*^n \to h_*^1 \to i_*^1$ will stay in the system as long as (i_*^ℓ, h_*^ℓ) continues to be a b-pair. If (i_*^ℓ, h_*^ℓ) continues to be a b-pair till round s' under $\succeq$, then the initial step is proved. Otherwise, there is a round $s'' \in \{s+1, ..., s'\}$ such that agent i_*^ℓ has brokerage right over h_*^ℓ at rounds $s, ..., s''-1$ but not at round s''. By C3's broker-to-heir transition property, $n = 2$ and $i_*^{\ell+1}$ owns h_*^ℓ at $\sigma^{s''}[\succ']$ because he owns $h_*^{\ell+1}$at both $\sigma^{s''-1}[\succ']$ and $\sigma^{s''}[\succ']$. As $i_*^{\ell+1}$ top preference is then h_*^ℓ, he will leave with it at s''. By C3's broker-to-heir transition property, agent i_*^ℓ will inherit $h_*^{\ell+1}$ at $s''+1$ and will be matched with it. This case ends the proof of the the inductive hypothesis for $r = s$.

Inductive step. Now, take any round $r > s$ such that $\sigma^r[\succ] - \sigma^{r-1}[\succ]$ is non-empty, and assume that the inductive hypothesis is true for all rounds up to $r - 1$. Consider agents and houses

$$h^1 \to i^1 \to h^2 \to ... \to h^n \to i^n \to h^1$$

that form a cycle at round r under $\succ$. Since all agents but i^* have same preferences in both profiles $\succ$ and $\succ'$, so do agents $i^1, ..., i^n$.

Let r' be the earliest round in which one of these agents is matched under $\succ'$, and let i^ℓ be an agent matched at r'. Recall that $\psi^{c,b}[\succ](i^\ell) = h^{\ell+1}$. The argument will be divided into ten claims, the first of which follows directly from the inductive hypothesis.

Claim 1. Suppose that a house $h \in \sigma^{r-1}[\succ]$ forms a cycle with at least two agents under $\succ$. Then:

- If agent $i \in I$ belongs to the cycle of house h under $\succ'$, then i belongs to the cycle of h under $\succ$.
- If house h' belongs to the cycle of h under $\succ'$, then h' belongs to the cycle of h under $\succ$.

Claim 2. Suppose $\nu = \sigma^{r-1}[\succ] \cup \sigma^{r''-1}[\succ']$ and agent i owns house h at ν, such that h belongs to a cycle at some round r'' under $\succ'$ and at round r under $\succ$. If j controls h at $\sigma^{r''-1}[\succ']$ and is unmatched at ν, then i is in one cycle with h at round r'' under $\succ'$.

Proof of Claim 2: If $i = j$, then the result is true. Assume $i \neq j$. Then j does not control h at ν. Thus, j brokers it at $\sigma^{r''-1}[\succ']$ and hence the cycle of h at round r'' under $\succ'$ contains some other agent j' and house h' that j' owns. By Claim 1, $j' \notin I_{\sigma^{r-1}[\succ]}$ and $h' \notin H_{\sigma^{r-1}[\succ]}$, and thus $j' \notin I_\nu$ and $h' \notin H_\nu$. By C1 and C3, j' owns h at ν, and thus $i = j'$. QED

Claim 3. Suppose that $n > 1$ and agents $j, i^1, ..., i^n$, and house h^1 are unmatched at $\nu = \sigma^{r-1}[\succ] \cup \sigma^{r''-1}[\succ']$, such that j and $h^{\ell+1}$ are part of a cycle matched at some round $r'' \leq r'$ under $\succ'$. If agent j controls $h^{\ell+1}$ at $\sigma^{r''-1}[\succ']$, then, under $\succ'$, $i^{\ell+1}$ and $h^{\ell+1}$ are both matched at round r'.

Proof of Claim 3: If j owns $h^{\ell+1}$ at $\sigma^{r''-1}[\succ']$ then C1 implies that both j and $i^{\ell+1}$ own $h^{\ell+1}$ at ν. Hence, $i^{\ell+1} = j$, and he is matched at r'' under $\succ'$. Because r' is the earliest round one of the agents $i^1, ..., i^n$ is matched, it must be that $r' = r''$ and the claim is true.

If j brokers $h^{\ell+1}$ at $\sigma^{r''-1}[\succ']$, then let σ_0 be a minimal submatching in

$$\left\{\sigma \in \mathcal{S} : \sigma^{r''-1}[\succ'] \subseteq \sigma \subseteq \nu\right\}$$

at which j is not a broker of $h^{\ell+1}$. Let $j' \neq j$ belong to the cycle of $h^{\ell+1}$ at round r'' under $\succ'$. Then j' is an owner of a house h' at $\sigma^{r''-1}[\succ']$. Because $n > 1$, Claim 1 gives $j' \notin I_{\sigma^{r-1}[\succ]}$ and $h' \notin H_{\sigma^{r-1}[\succ]}$. Thus, (j', h') is an o-pair at $\sigma_0 \subseteq \nu$. By C3, agent j' becomes the owner of $h^{\ell+1}$ at σ_0, and thus by C1, i.e., persistence of a ownership, $\left(j', h^{\ell+1}\right)$ is an o-pair at ν. Thus, $j' = i^{\ell+1}$ and he is matched at r'' in the same cycle as $h^{\ell+1}$ under $\succ'$. A fortiori, $r'' = r'$. QED

Claim 4. Assume i^ℓ is a $\sigma^{r-1}[\succ]$-owner. Under $\succ'$, agent i^ℓ will not be matched as long as house $h^{\ell+1}$ is unmatched. Furthermore, if i^ℓ and $h^{\ell+1}$ are matched in the same round then they are matched with each other.

Proof of Claim 4: House $h^{\ell+1}$ is agent i^ℓ's top choice among houses unmatched at $\sigma^{r-1}[\succ]$. By the inductive assumption, all houses matched before round r under $\succ$ are also matched with the same agents under $\succ$. Since $h^{\ell+1} = \psi^{c,b}[\succ]\left(i^\ell\right)$ is the top choice of houses remaining at round r under $\succ$, $\psi^{c,b}[\succ']\left(i^\ell\right)$ is weakly worse than $h^{\ell+1}$. Hence, at the round i^ℓ is matched he points to $h^{\ell+1}$ (and then is matched with it) or a worse house (and then $h^{\ell+1}$ was matched earlier). QED

Claim 5. Suppose that agents $j, i^1, ..., i^n$, and house $h^{\ell+1}$ are unmatched at $\nu = \sigma^{r-1}[\succ] \cup \sigma^{r''-1}[\succ']$, $i^{\ell+1}$ brokers $h^{\ell+1}$ at $\sigma^{r-1}[\succ]$, agent j controls $h^{\ell+1}$ at $\sigma^{r''-1}[\succ']$, such that agent j and house $h^{\ell+1}$ are part of a cycle matched at some round $r'' \leq r'$ under $\succ'$. Then, house $h^{\ell+1}$ is matched at round $r' = r''$ under $\succ'$.

Claim 6. Furthermore, the inductive hypothesis is true for round r or $i^{\ell+1}$ is matched at round r' under $\succ'$.

Proofs of Claim 5 and Claim 6: We will first prove Claim 5 and continue from that point on proving Claim 6. For convenience, assume $\ell = n$ that is $\ell + 1 = 1$. Since, i^1 is a broker when he was matched under $\succ$, $n > 1$, and thus, we can use Claim 1. Moreover, if $i^1 = j$, then i^1 controls h^1 at $\sigma^{r''-1}[\succ']$, and thus he is matched at round r'' under $\succ'$. Because r' is the earliest round one of the agents $i^1, ..., i^n$ is matched, it must be that $r' = r''$ and the claim is true. Hence, assume that $i^1 \neq j$.

Notice that

- If j controls h^1 at ν then i^1 does not.
- If j does not control h^1 at ν then j brokers it at $\sigma^{r''-1}[\succ']$, and thus, the cycle of h^1 contains some $\sigma^{r''-1}[\succ']$-owner j' and some house h' owned by j'. By Claim 1, $j' \notin I_{\sigma^{r-1}[\succ]}$ and $h' \notin H_{\sigma^{r-1}[\succ]}$, and thus j' and h' are not matched at ν. By C3, broker-to-heir transition is the only way j exists brokerage state, and thus, j' owns h^1 at ν, and hence i^1 cannot broker it at ν.

In either case, i^1 does not broker h^1 at ν while he brokers it at $\sigma^{r-1}[\succ]$. Thus, C3's broker-to-heir transition property implies that

$$n = 2, i^2 \text{ owns } h^1 \text{ at } \nu, \text{ and } i^1 \text{will own } h^2 \text{ if } i^2 \text{is matched with } h^1 \text{ at } \nu. \quad (1)$$

If $i^2 (= i^\ell) \neq j$, then Claim 2 implies that i^2 is in the cycle of h^1 that forms at round r'' under $\succ'$, and thus $r' = r''$.

If $i^2 (= i^\ell) = j$, Claim 4 implies that i^2 is matched at r'' under $\succ'$ in the cycle

$$h^1 \to i^2 \to h^1.$$

Thus, $r'' = r'$ and $h^1 (= h^{\ell+1})$ is matched at round r' under $\succ'$. That proves Claim 5.

In order to prove Claim 6, let r^1 be the time i^1 is matched, and r^2 be the time h^2 is matched under $\succ'$. By Claim 7 (whose proof depends on Claim 5 but not on Claim 6), $r^2 \leq r^1$. Let j^2 be the agent controlling h^2 at $\sigma^{r^2-1}[\succ']$. By Claim 1, $j^2 \notin I_{\sigma^{r-1}[\succ]}$. Let

$$\nu' = \sigma^{r-1}[\succ] \cup \sigma^{r^2-1}[\succ'].$$

If $r^2 > r'$ then i^2 is matched to h^1 at $\nu' \supseteq \sigma^{r^2-1}[\succ'] \supseteq \sigma^{r'}[\succ']$. By Statement in (1), i^1 owns h^2 at ν'. By Claim 2, i^1 is in one cycle with h^2 at r^2.

If $r^2 \leq r'$ then i^2, the $\sigma^{r-1}[\succ]$-owner of h^2, is unmatched at $\sigma^{r^2-1}[\succ']$ and hence, owns h^2 at ν'. By Claim 2, $i^2 \left(= i^\ell\right)$ is in one cycle with $h^2 \left(= h^\ell\right)$ at r^2 under $\succ'$. That means that $r^2 = r'$. Let i be the agent controlling h^1 at $\sigma^{r'-1}[\succ']$. Since the cycle of h^1 contains i^1 and i^2, and i^2 gets h^1, hence $i \neq i^2$. Because, i^2 owns h^1 and h^2 at ν', C3 implies that i inherits h^2 if i^2 is matched with h^1 at ν'. By Statement in (1), i^1 inherits h^2 if i^2 is matched with h^1 at ν, and hence also at ν'. Thus, $i^2 \left(= i^\ell\right) = i$ belongs to the cycle of $h^1 \left(= h^{\ell+1}\right)$ under $\succ'$. QED

Claim 7. Assume i^ℓ is a $\sigma^{r-1}[\succ]$-broker. Under $\succ'$, agent i^ℓ will not be matched as long as house $h^{\ell+1}$ is unmatched. Furthermore, if i^ℓ and $h^{\ell+1}$ are matched in the same round, then they are matched with each other.

Proof of Claim 7: Notice that $n > 1$ and for notational convenience assume that $\ell = 1$. By the inductive assumption, There exists r^* such that $\sigma^{r-1}[\succ] \subseteq \sigma^{r^*}[\succ']$. Thus, if (i^1, h^2) does not satisfy the claim then the top preference of i^1 must be h^1, the house he brokers at $\sigma^{r-1}[\succ]$, and i^1 must get h^1 under $\succ'$. Let j be the agent controlling h^1 at $\sigma^{r'-1}[\succ']$. Notice that j is matched in the same cycle as h^1 at round r' under $\succ'$. Since $h^1 \notin H_{\sigma^{r-1}[\succ]}$ and $n > 1$, Claim 1 implies that $j \notin I_{\sigma^{r-1}[\succ]}$. Thus, agents $j, i^1, ..., i^n$ and house h^1 are unmatched at the submatching $\nu = \sigma^{r-1}[\succ] \cup \sigma^{r''-1}[\succ']$. By Claim 5, h^1 is matched at r', and by Claim 4, i^n gets h^1, a contradiction. QED

Claim 8. If $n = 1$ then either $h^1 \to i^1 \to h^1$ form a cycle under $\succ'$, or there exists an agent j and a house h such that $h \to i^1 \to h^1 \to j \to h$ form a cycle under $\succ'$.

Proof of Claim 8: Claim 2 implies that h^1 is matched at round $r'' \leq r'$ when preferences are $\succ'$. Let j be the agent controlling h^1 at $\sigma^{r''-1}[\succ']$. Notice that j is matched in the same cycle as h^1 at round r'' under $\succ'$. Two cases are possible about j:

Case 1. $j \in I_{\sigma^{r-1}[\succ]}$: Then $j \neq i^1$ and the inductive assumption and $h^1 \notin H_{\sigma^{r-1}[\succ]}$ imply that

- j is matched at round r under $\succ$ in a cycle $h \to j \to h$ (for some house $h \neq h^1$), and
- there exists agent i such that the cycle $h \to i \to h^1 \to j \to h$ is matched at round r'' under $\succ'$.

To finish the proof of the current case, it remains to be shown that $i = i^1$.

- If $i \in I_{\sigma^{r-1}[\succ]}$ then the inductive assumption implies that i is matched with h^1 under $\succ'$, and hence $i = i^1$.
- If $i \notin I_{\sigma^{r-1}[\succ]}$ then i and i^1 are unmatched at $\sigma^{r-1}[\succ] \cup \sigma^{r''-1}[\succ']$, and by C1, i.e., persistence of o-pairs, i^1 owns h^1 at this submatching. Notice that j is an owner of h at $\sigma^{r-2}[\succ]$, and hence at $\sigma^{r-2}[\succ] \cup \sigma^{r''-1}[\succ']$. Thus, i must have been a broker of h at $\sigma^{r''-1}[\succ']$ and stopped being a broker at a submatching σ between $\sigma^{r''-1}[\succ']$ and $\sigma^{r-2}[\succ] \cup \sigma^{r''-1}[\succ']$. Because there might be only one broker at each submatching, j is an owner of h^1 at $\sigma^{r''-1}[\succ']$. Thus j is an owner at σ, and inherits h when i loses the broker status. When j is matched with h, the broker-to-heir transition rules (C3) imply that i becomes the owner of h^1. Hence, i is the owner of h^1 at $\sigma^{r-1}[\succ] \cup \sigma^{r''-1}[\succ']$ as is i^1. Thus, $i = i^1$.

Case 2. $j \notin I_{\sigma^{r-1}[\succ]}$: Then, agents j, i^1 are unmatched at the submatching $\nu = \sigma^{r-1}[\succ] \cup \sigma^{r''-1}[\succ']$, and by C1, i.e., persistence of o-pairs, i^1 owns h^1 at this submatching. Hence, either

- $j = i^1$ controls h^1 at $\sigma^{r''-1}[\succ']$, or
- $j \neq i^1$ is a broker of h^1 at $\sigma^{r''-1}[\succ']$, and loses the brokerage right at some submatching σ between $\sigma^{r''-1}[\succ']$ and $\sigma^{r-1}[\succ] \cup \sigma^{r''}[\succ']$.

In the former subcase, i^1 is matched at r'' as h^1 is matched under $\succ'$. Thus, $r'' = r'$, and a fortiori i^1 is matched with h^1 $\left(= i^\ell\right)$ and hence owns h^1 at r'. The claim is then proved.

In the latter subcase, let $j' \neq j$ be an agent matched in the same cycle as h^1 at round r'' under $\succ'$. Then j' is an owner of a house h' at $\sigma^{r''-1}[\succ']$.

We have $j' \notin I_{\sigma^{r-1}[\succ]}$ and $h' \notin H_{\sigma^{r-1}[\succ]}$, as otherwise the inductive assumption and $h^1 \notin H_{\sigma^{r-1}[\succ]}$ would imply that j' is matched at $\sigma^{r-1}[\succ]$ before round r under $\succ$ in a cycle $h' \to j' \to h'$, and there would exist an agent i such that the cycle $h' \to i \to h^1 \to j' \to h'$ is matched at r'' under $\succ'$. The inductive assumption would further imply that h^1 is matched to i at $\succ$ and thus $i^1 = i$. Since j is in the cycle of h^1 and $j \neq j'$ and $j \neq i^1$, we would obtain a contradiction showing that $j' \notin I_{\sigma^{r-1}[\succ]}$.

Thus j' and h' are unmatched at $\sigma^{r-1}[\succ] \cup \sigma^{r''-1}[\succ']$, and hence they are unmatched at σ. Thus, C3's broker-to-heir transition property implies that agent j' is the owner of h^1 at σ, and by persistence of o-pairs through C1, (j', h^1) is an o-pair at $\sigma^{r-1}[\succ] \cup \sigma^{r''-1}[\succ']$. Thus, $i^1 = j'$ and he is matched with h^1 at $r'' = r'$. Thus, at round r' under $\succ'$ the cycle in which i^1 is matched is $h' \to i^1 \to h^1 \to j \to h'$. QED

Claim 9. Suppose $n > 1$. If $i^{\ell+1}$ is $\sigma^{r-1}[\succ]$-owner, then i^{ℓ} is matched with $h^{\ell+1}$ at round r' under $\succ$, and $i^{\ell+1}$ is also matched at round r' under $\succ'$.

Proof of Claim 9: By Claim 2, house $h^{\ell+1}$ is matched at round $r'' \leq r'$ under $\succ'$. Let j be the owner or broker of the house $h^{\ell+1}$ at $\sigma^{r''-1}[\succ']$. Notice that j is matched in the same cycle as $h^{\ell+1}$ at round r'' under $\succ'$. Since $h^{\ell+1} \notin H_{\sigma^{r-1}[\succ]}$ and $n > 1$, Claim 1 implies that $j \notin I_{\sigma^{r-1}[\succ]}$. Thus, agents $j, i^1, ..., i^n$ are unmatched at the submatching $\sigma^{r-1}[\succ] \cup \sigma^{r''-1}[\succ']$, and Claim 3 yields that $i^{\ell+1}$ is matched at $r'' = r'$ and then Claim 2 shows that i^{ℓ} is matched with $h^{\ell+1}$. QED

Claim 10. If $i^{\ell+1}$ is a $\sigma^{r-1}[\succ]$-broker, then either the inductive hypothesis is true or i^{ℓ} is matched with $h^{\ell+1}$ at round r' under $\succ'$, and $i^{\ell+1}$ is also matched at round r' under $\succ'$.

Proof of Claim 10: For convenience let us assume that $\ell = n$ and $\ell + 1 = 1$. Agent i^n is a $\sigma^{r-1}[\succ]$-owner because $n > 1$ and i^1 is a $\sigma^{r-1}[\succ]$-broker. By Claim 2, house h^1 is matched at round $r'' \leq r'$ under $\succ'$. Let j be the owner or broker of the house at $\sigma^{r''-1}[\succ']$. Notice that j is matched in the same cycle as h^1 at round r'' under $\succ'$. Since $h^1 \notin H_{\sigma^{r-1}[\succ]}$ and $n > 1$, Claim 1 implies that $j \notin I_{\sigma^{r-1}[\succ]}$. Thus, agents $j, i^1, ..., i^n$ are unmatched at the submatching $\sigma^{r-1}[\succ] \cup \sigma^{r''-1}[\succ']$. Claim 5 yields $r'' = r'$ and thus Claim 4 shows that i^n is matched with h^1 under $\succ'$. Claim 6 ends the proof. QED

Claim 8 proves the inductive hypothesis for cycles of length $n = 1$ and Claim 9 and Claim 10 applied iteratively prove the hypothesis for cycles of length $n > 1$. This ends the proof of the theorem. **QED**

B Appendix: Proof of Theorem 4 (Implementation Result)

Let φ be a coalitionally strategy-proof and Pareto-efficient mechanism (fixed throughout Appendix B). We are to prove that φ may be represented as a TCBO mechanism. We will first construct the candidate control rights structure (c, b) and then show that the induced TCBO mechanism $\psi^{c,b}$ is equivalent to φ. We will state some preparatory definitions and lemmata first:

In the construction of the control rights structure we will use the following notation (partially introduced already in Example 3). Let $\sigma \in \mathcal{S} - \mathcal{M}$, $n \geq 0$ and $h^1, h^2, ..., h^n \in H - H_\sigma$, and $i \in I$.

$\mathbf{P}_i\left[\sigma, h^1, ..., h^n\right]$ is the set of preferences $\succ_i$ of agent i such that

- if $i \in I_\sigma$, then

$$\sigma(i) \succ_i g \quad \text{for all } g \in H - \{\sigma(i)\},$$

- if $i \in I - I_\sigma$, then

$$h^1 \succeq_i h^2 \succeq ... \succeq_i h^n \succ_i g \succ_i g' \text{ for all } g \in H - H_\sigma - \left\{h^1, ..., h^n\right\} \text{ and all } g' \in H_\sigma.$$

That is, if i is not matched in submatching σ, $\mathbf{P}_i\left[\sigma, h^1, ..., h^n\right]$ is the set of preferences that rank h^1,...,h^n in order over the remaining houses unmatched under σ, and rank those over the houses matched under σ; otherwise, $\mathbf{P}_i\left[\sigma, h^1, ..., h^n\right]$ is the set of preferences that rank agent i's match under σ over all other houses (observe that $\mathbf{P}_i[\varnothing] \equiv \mathbf{P}_i$).

$\mathbf{P}\left[\sigma, h^1, ..., h^n\right] \subseteq \mathbf{P}$ is the Cartesian product of $\mathbf{P}_i\left[\sigma, h^1, ..., h^n\right]$ over all $i \in I$. We define

$$\mathbf{P}^*[\sigma, h] = \cup_{h' \in (H - H_\sigma) - \{h\}} \mathbf{P}\left[\sigma, h, h'\right],$$

i.e., the set of preference profiles generated by $\mathbf{P}[\sigma, h]$ that rank the same house as the second choice across all agents unmatched under σ.

When σ is fixed, we will occasionally write $\langle h^1, ..., h^n, ...\rangle$ instead of $\mathbf{P}_i\left[\sigma, h^1, ..., h^n\right]$.

We are ready to introduce some new terminology for the mechanism φ that is similar to the control rights structure terminology of the TCBO mechanisms. To distinguish the two classes defined for TCBO and φ, we will suffix these new definitions with *:

A house $h \in H$ is an **o-house* at** $\sigma \in \mathcal{S}$ if it is unmatched at σ and the agent $\varphi[\succ]^{-1}(h)$ is constant across all $\succ \in \mathbf{P}^*[\sigma, h]$. We refer to the agent $\varphi[\succ]^{-1}(h)$ as the **owner* of** h **at** σ. We refer to the pair $\left(\varphi[\succ]^{-1}(h), h\right)$ as an **o-pair* at** σ. We say that (i, h) is a **strong o-pair* at** σ if for all $\succ \in \mathbf{P}[\sigma, h]$, we have $\varphi[\succ](i) = h$. Observe that strong ownership* implies ownership*.

A house $f \in H$ is a **b-house* at** $\sigma \in \mathcal{S}$ if it is unmatched at σ and there exist some $\succ$ and $\succ' \in \mathbf{P}^*[\sigma, f]$ such that $\varphi[\succ]^{-1}(f) \neq \varphi[\succ']^{-1}(f)$. Agent k is the **broker* of** f **at** σ if f is a σ-b-house* and for all $\succ \in \mathbf{P}^*[\sigma, f]$ house $\varphi[\succ](k)$ is the second choice of k in $\succ_k$. We refer to the pair (k, f) as a **b-pair* at** σ. Pair (k, f) is a **strong b-pair* at** σ if f is a b-house* at σ and for all $\succ \in \mathbf{P}[\sigma, f]$ such that $\succ_k \in \langle f, h, ... \rangle$, we have $\varphi[\succ](k) = h$. Observe that strong brokerage* implies brokerage*.

In Lemma 8 and Corollary 3 below, we will prove that strong ownership* and ownership*, and strong brokerage* and brokerage* are equivalent to each other, respectively. Notice that if φ is a TCBO mechanism and i is an owner at σ then i is an (strong) owner* at σ, similarly for (strong) broker*. Thus, (strong) owners* and (strong) brokers* are the *candidate* owners and brokers in the TCBO mechanism that we will construct. We will show that the starred terms can be used to determine a compatible control rights structure (c, b) and a TCBO mechanism $\psi^{c,b}$. The proof of Theorem 4 will be finished after we show that $\varphi = \psi^{c,b}$.

Two lemmata proved in Pápai (2000) will be useful. Following Pápai (2000), we say that j **envies** i **at** $\succ$ if

$$\varphi[\succ](i) \succ_j \varphi[\succ](j).$$

Lemma 3 *(Pápai 2000) For all $i, j \in I$, all $\succ \in \mathbf{P}$, and all $\succ_j^* \in \mathbf{P}_j$, if j envies i at $\succ$ and $\varphi\left[\succ_j^*, \succ_{-j}\right](i) \neq \varphi[\succ](i)$, then*

$$\varphi[\succ](i) \succ_i \varphi\left[\succ_j^*, \succ_{-j}\right](i).$$

Lemma 4 *(Pápai 2000) For all $i, j \in I$, all $\succ \in \mathbf{P}$, and all $\succ_j^* \in \mathbf{P}_j$, if j envies i at $\succ$ and $\varphi\left[\succ_j^*, \succ_{-j}\right](i) \neq \varphi[\succ](i)$, then there exists $\succ_i^* \in \mathbf{P}_i$ such that*

$$\varphi\left[\succ_i^*, \succ_j^*, \succ_{-\{i,j\}}\right](i) = \varphi[\succ](j).$$

The following is an immediate corollary of strategy-proofness and Lemma 4:

Corollary 2 *For all* $i, j \in I$*, all* $\succ \in \mathbf{P}$*, and all* $\succ_j^* \in \mathbf{P}_j$*, if* j *envies* i *at* $\succ$ *and* $\varphi\left[\succ_j^*, \succ_{-j}\right](i) \neq \varphi[\succ](i)$*, then*

$$\varphi\left[\succ_j^*, \succ_{-j}\right](i) \succeq_i \varphi[\succ](j).$$

To show that the starred terms introduced above determine a compatible control rights structure, we will prove a sequence of lemmata. We start with lemmata determining relations between ownership* and brokerage* at a fixed submatching $\sigma \in \mathcal{S} - \mathcal{M}$.

B.1 "Intratemporal" Lemmata (For Fixed Submatching)

Lemma 5 *Let* $\sigma \in \mathcal{S} - \mathcal{M}$*. For all* $i \in I_\sigma$ *and all* $h \in H - H_\sigma$*,*

$$\varphi[\succ](i) = \sigma(i) \text{ for all } \succ \in \mathbf{P}[\sigma, h].$$

Proof of Lemma 5. Suppose that an agent in $i \in I_\sigma$ does not get $\sigma(i)$ at $\varphi[\succ]$. Then we can create a matching by assigning all agents in $I - I_\sigma$ that get a house in H_σ a house in $H - H_\sigma$ that was assigned to an agent in I_σ, all other agents j in $I - I_\sigma$ the house $\varphi[\succ](j)$, and all agents j in I_σ the house $\sigma(j)$. Since each agent in $I - I_\sigma$ ranks houses in H_σ lower than houses in $H - H_\sigma$ and each agent in I_σ ranks his σ-house as his first choice, this new matching Pareto-dominates $\varphi[\succ]$, contradicting φ is Pareto-efficient. **QED**

Lemma 6 *Let* $\sigma \in \mathcal{S} - \mathcal{M}$ *and* $f, h \in H - H_\sigma$*. Then there exists some agent* $i \in I - I_\sigma$ *such that* $\varphi[\succ](i) = f$ *for all* $\succ \in \mathbf{P}[\sigma, f, h]$*.*

Proof of Lemma 6. Suppose not. Let $\succ, \succ' \in \mathbf{P}[\sigma, f, h]$ be such that $\varphi[\succ](i) = f$ and $\varphi[\succ'](i') = f$ for some $i' \neq i$.

Without loss of generality, we assume that $\succ$ and $\succ'$ differ only in preferences of a single agent $j \in I - I_\sigma$. Its argument is as follows: Suppose that $\succ'$ and $\succ$ differ in preferences of agents $i_1, i_2, ..., i_n$. We can obtain $\succ'$ from $\succ$ by replacing $\succ_{i_m}$ with $\succ'_{i_m}$ one at a time for all $m = 1, 2, ..., n$. At some m, we have $\varphi\left[\succ'_{\{i_1,...,i_{m-1}\}}, \succ_{-\{i_1,...,i_{m-1}\}}\right](i) = f$ and $\varphi\left[\succ'_{\{i_1,...,i_m\}}, \succ_{-\{i_1,...,i_m\}}\right](i) \neq f$. At this point, we relabel $\left[\succ'_{\{i_1,...,i_{m-1}\}}, \succ_{-\{i_1,...,i_{m-1}\}}\right]$ as $\succ$ and $\left[\succ'_{\{i_1,...,i_m\}}, \succ_{-\{i_1,...,i_m\}}\right]$ as $\succ'$. Thus, $\succ$ and $\succ'$ differ only in preferences of a single agent $j \equiv i_m$.

Let $g = \varphi[\succ](j)$ and $g' = \varphi[\succ'](j)$. By non-bossiness, $g \neq g'$. By strategy-proofness, $j \notin \{i, i'\}$, and hence, $f \notin \{g, g'\}$.

Claim 1: We have $h \neq g$. Moreover, without loss of generality, we can choose $\succ_i \in \langle f, h, g, ... \rangle$ so that $\varphi[\succ'](i) = g$.

Proof of Claim 1:

- Originally, we have $\succ_i \in \langle f, h, ..., g, ... \rangle$. By Corollary 2, $\varphi[\succ'](i) \succeq_i g$ and yet, $\varphi[\succ'](i) \neq f$. Thus, by strategy-proofness for i, for all $\succ''_i \in \langle f, h, g, ... \rangle$, $\varphi\left[\succ''_i, \succ'_{-i}\right](i) \in \{g, h\}$. Moreover, by again strategy-proofness for i, $\varphi[\succ''_i, \succ_{-i}](i) = f$. Non-bossiness for i implies that $\varphi[\succ''_i, \succ_{-i}] = \varphi[\succ]$, and in particular, $\varphi[\succ''_i, \succ_{-i}](j) = g$. Since j envies i at $[\succ''_i, \succ_{-i}]$ and changes his allocation between $\succ$ and $\succ'$, by Lemma 3, he cannot envy i at $\left[\succ''_i, \succ'_{-i}\right]$. Since $\varphi\left[\succ''_i, \succ'_{-i}\right](j) \prec'_j h$ by strategy-proofness for j from $[\succ''_i, \succ_{-i}]$ to $\left[\succ''_i, \succ'_{-i}\right]$, we have $\varphi\left[\succ''_i, \succ'_{-i}\right](i) = g$ and $g \neq h$.

- Since $\varphi\left[\succ''_i, \succ'_{-i}\right] \neq \varphi[\succ''_i, \succ_{-i}]$ and only j's preferences are different between the two profiles, by non-bossiness for j, $\varphi\left[\succ''_i, \succ'_{-i}\right](j) \neq \varphi[\succ''_i, \succ_{-i}](j) = g$. Thus, $\succ''_i$ satisfies all the properties expected from $\succ_i$ above, and hence, we can relabel the preference $\succ''_i$ as $\succ_i$ and house $\varphi\left[\succ''_i, \succ'_{-i}\right](j)$ as g'. Let $i' = \varphi\left[\succ''_i, \succ'_{-i}\right]^{-1}(f)$ be relabeled as well. QED

By Claim 1, without loss of generality, we assume $\succ_i \in \langle f, h, g, ... \rangle$.

Moreover, by strategy-proofness and non-bossiness for j, without loss of generality, we further assume that

$$\succ_j \in \langle f, h, g, g', ... \rangle \text{ and } \succ'_j \in \langle f, h, g', g, ... \rangle,$$

and that the only difference between $\succ_j$ and $\succ'_j$ is in relative ranking of g and g'.

By Pareto efficiency, an agent from $I - I_\sigma$ is assigned h at $\succ$. Let $k = \varphi[\succ]^{-1}(h)$.

Define

$$\succ^*_k \in \langle f, g, h, ... \rangle,$$
$$\succ^*_i \in \langle h, f, ... \rangle, \text{ and}$$
$$\succ^*_{i'} \in \langle h, f, ... \rangle.$$

Claim 2. If $\succ_k \in \langle h, f, g, ...\rangle$, then $\varphi[\succ'](k) = h$.

Proof of Claim 2. Suppose that $\succ_k \in \langle h, f, g, ...\rangle$. Since agent j envies k at $\succ$ and $\varphi[\succ](j) = g$, Corollary 2 and strategy-proofness of φ imply that k gets at least g at $\succ'$. Suppose on the contrary of the claim that $\varphi[\succ'](k) \neq h = \varphi[\succ](k)$. Thus, by Lemma 3, j cannot envy k also at $\succ'$. Hence, $\varphi[\succ'](k) \notin \{f, h\}$. However, we showed that $\varphi[\succ'](i) = g$ and we also know that $i \neq k$. Thus, $\varphi[\succ'](k) \prec'_k g$ contradicting $\varphi[\succ'](k) \succeq'_k g$. We showed that j cannot change the allocation of k between $\succ$ and $\succ'$, and thus, $\varphi[\succ'](k) = \varphi[\succ](k) = h$. QED

Thus, by Claim 2, and by strategy-proofness and non-bossiness for k, without loss of generality, we further assume that

$$\succ_k \in \langle f, h, g, g'...\rangle .$$

Claim 3. $\varphi[\succ_i^*, \succ_{-i}](i) = h$ and $\varphi[\succ_i^*, \succ_{-i}](j) = g$.

Proof of Claim 3.

- By strategy-proofness, $\varphi[\succ_i^*, \succ_{-i}](i) \succeq_i^* f$. Everybody else in $I - I_\sigma$ ranks f over h. Thus, by Lemma 5 and Pareto efficiency, i should get h at $[\succ_i^*, \succ_{-i}]$.
- By Maskin monotonicity regarding i, $\varphi\left[\succ_i^*, \succ'_{-i}\right] = \varphi[\succ']$. Thus, j gets g' at $\left[\succ_i^*, \succ'_{-i}\right]$. By strategy-proofness for j, agent j gets at least g' and no house better than g at $[\succ_i^*, \succ_{-i}]$ (recall that between $\succ_{-i}$ and $\succ'_{-i}$ only j changes his preferences). Suppose j gets g' at $[\succ_i^*, \succ_{-i}]$. Then, by Maskin monotonicity, we have $\varphi\left[\succ_i^*, \succ'_{-i}\right] = \varphi[\succ_i^*, \succ_{-i}]$. In particular, $\varphi\left[\succ_i^*, \succ'_{-i}\right](i) = \varphi[\succ_i^*, \succ_{-i}](i) = h$. Again by strategy-proofness and non-bossiness for i, $\varphi\left[\succ_i^*, \succ'_{-i}\right] = \varphi[\succ']$. On the other hand, we have $\varphi[\succ'](k) = h$ by Claim 2, contradicting $\varphi\left[\succ_i^*, \succ'_{-i}\right](i) = \varphi[\succ'](i) = h$. Therefore, $\varphi[\succ_i^*, \succ_{-i}](j) = g$.

QED

Claim 4. $\varphi[\succ_k^*, \succ_{-k}](k) = \varphi\left[\succ_k^*, \succ'_{-k}\right](k) = g$.

Proof of Claim 4. Recall that $\succ_k^* \in \langle f, g, h, ...\rangle$. By strategy-proofness, since k gets h at $\succ$, agent k cannot get f and gets at least h at $[\succ_k^*, \succ_{-k}]$. Thus, k gets h or g at $[\succ_k^*, \succ_{-k}]$. Everybody else in $I - I_\sigma$ ranks h over g. Thus, by Lemma 5 and Pareto efficiency, agent k should get g at $[\succ_k^*, \succ_{-k}]$.

By Claim 2, agent k gets h at $\succ'$. A symmetric argument to the one above shows that k gets g at $\left[\succ_k^*, \succ'_{-k}\right]$. QED

Claim 5. $\varphi\left[\succ_k^*, \succ'_{-k}\right] = \varphi\left[\succ_k^*, \succ_{-k}\right]$.

Proof of Claim 5. These two profiles, $\left[\succ_k^*, \succ'_{-k}\right]$ and $\left[\succ_k^*, \succ_{-k}\right]$, only differ in preferences of agent j who ranks g above g' at $\succ_j$ and the other way at $\succ'_j$. By Claim 4, j does not get g at $\left[\succ_k^*, \succ_{-k}\right]$. Maskin monotonicity implies $\varphi\left[\succ_k^*, \succ'_{-k}\right] = \varphi\left[\succ_k^*, \succ_{-k}\right]$. QED

Claim 6. $\{\varphi\left[\succ_k^*, \succ_{-k}\right](i), \varphi\left[\succ_k^*, \succ_{-k}\right](i')\} = \{f, h\}$.

Proof of Claim 6. Agent k envies agent i at $\succ$. Thus, by Corollary 2, agent i gets at least $h = \varphi[\succ](k)$ at $\left[\succ_k^*, \succ_{-k}\right]$. Hence $\varphi\left[\succ_k^*, \succ_{-k}\right](i) \in \{f, h\}$. By Claim 2, $\varphi[\succ'](k) = h$. The symmetric argument as above shows that $\varphi\left[\succ_k^*, \succ'_{-k}\right](i') \in \{f, h\}$. Furthermore, Claim 5 implies that $\varphi\left[\succ_k^*, \succ_{-k}\right](i') = \varphi\left[\succ_k^*, \succ'_{-k}\right](i')$. Thus, $\varphi\left[\succ_k^*, \succ_{-k}\right](i)$ and $\varphi\left[\succ_k^*, \succ_{-k}\right](i')$ are different and both belong to $\{f, h\}$. QED

Claim 7. $\varphi\left[\succ_k^*, \succ_{-k}\right](i) = f$ and $\varphi\left[\succ_k^*, \succ_{-k}\right](i') = h$.

Proof of Claim 7. Suppose not for an indirect argument. Then, Claim 6 implies that $\varphi\left[\succ_k^*, \succ_{-k}\right](i) = h$ and $\varphi\left[\succ_k^*, \succ_{-k}\right](i') = f$. By Maskin monotonicity, $\varphi\left[\succ_k^*, \succ_{-k}\right] = \varphi\left[\succ_k^*, \succ_i^*, \succ_{-\{k,i\}}\right]$. By this equivalence and Claim 4, we have $\varphi\left[\succ_k^*, \succ_i^*, \succ_{-\{k,i\}}\right](k) = g$. By strategy-proofness, agent k gets at least g and not f at $\left[\succ_i^*, \succ_{-i}\right]$. This contradicts Claim 3. QED

Claim 8. $\varphi\left[\succ_{\{k,i\}}^*, \succ_{-\{k,i\}}\right](i) = \varphi\left[\succ_{\{k,i\}}^*, \succ'_{-\{k,i\}}\right](i) = h$ and $\varphi\left[\succ_{\{k,i\}}^*, \succ'_{-\{k,i\}}\right](k) = g$.

Proof of Claim 8.

- By strategy-proofness for i and Claim 7, $\varphi\left[\succ_k^*, \succ_i^*, \succ_{-\{k,i\}}\right](i) \succeq_i^* f$, implying $\varphi\left[\succ_k^*, \succ_i^*, \succ_{-\{k,i\}}\right](i) \in \{h, f\}$. Since only i ranks h above f at $\left[\succ_k^*, \succ_i^*, \succ'_{-\{k,i\}}\right]$ among all agents in $I - I_\sigma$, Pareto-efficiency and Lemma 5 imply that $\varphi\left[\succ_k^*, \succ_i^*, \succ_{-\{k,i\}}\right](i) = h$.

  By Claim 5, the above argument is also true for $\left[\succ_k^*, \succ_i^*, \succ'_{-\{k,i\}}\right]$ instead of $\left[\succ_k^*, \succ_i^*, \succ_{-\{k,i\}}\right]$, and thus, $\varphi\left[\succ_k^*, \succ_i^*, \succ'_{-\{k,i\}}\right](i) = h$.

- Three cases are possible regarding $\varphi\left[\succ_k^*, \succ_i^*, \succ'_{-\{k,i\}}\right](k)$:

  Case 1. $\varphi\left[\succ_k^*, \succ_i^*, \succ'_{-\{k,i\}}\right](k) = f$: Since $\varphi[\succ'](k) = h$ by Claim 2 and $\varphi[\succ'](i) = g$ by Claim 1, agents k and i can jointly manipulate from $\succ'$ to $\left[\succ_k^*, \succ_i^*, \succ'_{-\{k,i\}}\right]$ and they strictly benefit from this manipulation with respect to $\succ'$, contradicting coalitional strategy-proofness.

  Case 2. $\varphi\left[\succ_k^*, \succ_i^*, \succ'_{-\{k,i\}}\right](k) \prec_k g$: Then by Maskin monotonicity for k, $\varphi\left[\succ_i^*, \succ'_{-i}\right] = \varphi\left[\succ_k^*, \succ_i^*, \succ'_{-i}\right]$. Thus, $\varphi\left[\succ_i^*, \succ'_{-i}\right](i) = h$. Since by Claim 1, $\varphi[\succ'](i) = g$, agent i lies and benefits at $\succ'$, contradicting strategy-proofness.

  Case 3. $\varphi\left[\succ_k^*, \succ_i^*, \succ'_{-\{k,i\}}\right](k) = g$: By the above analysis, this should be true.

QED

Claim 9. $\varphi\left[\succ_{\{i,i'\}}^*, \succ_{-\{i,i'\}}\right](i') \notin \{f, h\}$ and thus, $\varphi\left[\succ_{\{i,i'\}}^*, \succ_{-\{i,i'\}}\right] = \varphi\left[\succ_i^*, \succ_{-i}\right]$.

Proof of Claim 9. Suppose that $\varphi\left[\succ_{\{i,i'\}}^*, \succ_{-\{i,i'\}}\right](i') = h$. Then by strategy-proofness and non-bossiness for i, $\varphi\left[\succ_{i'}^*, \succ_{-i'}\right] = \varphi\left[\succ_{\{i,i'\}}^*, \succ_{-\{i,i'\}}\right]$, and in particular, $\varphi\left[\succ_{i'}^*, \succ_{-i'}\right](i') = h$. By strategy-proofness for i', $\varphi[\succ](i') \succeq_{i'} h$, contradicting $\varphi[\succ](i) = f$, $\varphi[\succ](k) = h$, and thus, $\varphi[\succ](i') \prec_{i'} h$.

Suppose that $\varphi\left[\succ_{\{i,i'\}}^*, \succ_{-\{i,i'\}}\right](i') = f$. By Claim 8 and strategy-proofness for k, $\varphi\left[\succ_{\{i,i'\}}^*, \succ_{-\{i,i'\}}\right](i) = h$. If j submitted $\succ'_j$ instead of $\succ_j$, the situation would be totally symmetric in all above claims by switching i with i' and g with g' in k's and j's preferences. Thus, the same claims would imply $\varphi\left[\succ_{\{i,i'\}}^*, \succ'_{-\{i,i'\}}\right](i') = h$ and $\varphi\left[\succ_{\{i,i'\}}^*, \succ'_{-\{i,i'\}}\right](i) = f$. Observe that at $\left[\succ_{\{i,i'\}}^*, \succ_{-\{i,i'\}}\right]$, j envies i'. By submitting $\succ'_j$, agent j makes i' better off, contradicting Lemma 3. Thus, we showed that $\varphi\left[\succ_{\{i,i'\}}^*, \succ_{-\{i,i'\}}\right](i') \neq f$, either.

Since $\succ_{i'}^*$ only switches h's ranking with f's with respect to $\succ_{i'}$, Claim 8, strategy-proofness and non-bossiness for i' imply that $\varphi\left[\succ_{\{i,i'\}}^*, \succ_{-\{i,i'\}}\right] = \varphi\left[\succ_i^*, \succ_{-i}\right]$. QED

We complete the proof of the lemma as follows:

By Claim 2 and by the symmetry among agents i and i' and houses g and g', we can state and prove Claims 3-9 also by switching houses g' and g, and agents i' and i with one and other in all of the statements and proofs.[12] Thus, for all $\succ_k^{*\prime} \in \langle f, h, g', g ... \rangle$, by Claim 5 $\varphi\left[\succ_k^{*\prime}, \succ'_{-k}\right] = \varphi\left[\succ_k^{*\prime}, \succ_{-k}\right]$, by Claim

[12] Although $\succ_i$ and $\succ_{i'}$ are not totally symmetric regarding g and g', i.e. $\succ_i$ has g as the third choice, while g' is ranked as the third or worse at $\succ_{i'}$, Claims 3-9 symmetrically hold. This is true because, we only use the fact that $\succ_i$ has g as the third choice in proving Claims 1 and 2.

4 $\varphi\left[\succ_k^{*\prime}, \succ_{-k}'\right](k) = \varphi\left[\succ_k^{*\prime}, \succ_{-k}\right](k) = g'$, and by Claim 7 $\varphi\left[\succ_k^{*\prime}, \succ_{-k}\right](i') = f$ and $\varphi\left[\succ_k^{*\prime}, \succ_{-k}\right](i) = h$. Hence, by strategy-proofness and non-bossiness for i, we obtain $\varphi\left[\succ_k^{*\prime}, \succ_i^*, \succ_{-\{k,i\}}\right] = \varphi\left[\succ_k^{*\prime}, \succ_{-k}\right]$.

On the other hand, by Claim 3 $\varphi\left[\succ_i^*, \succ_{-i}\right](i) = h$ and $\varphi\left[\succ_i^*, \succ_{-i}\right](j) = g$. Thus, $\varphi\left[\succ_i^*, \succ_{-i}\right](k) = f$ or $\varphi\left[\succ_i^*, \succ_{-i}\right](k) \prec_k g$. We consider these two cases separately:

Case 1. $\varphi\left[\succ_i^*, \succ_{-i}\right](k) = f$: We have $\varphi\left[\succ_k^{*\prime}, \succ_{-k}\right](k) = g'$ and $\varphi\left[\succ_k^{*\prime}, \succ_i^* \succ_{-\{k,i\}}\right] = \varphi\left[\succ_k^{*\prime}, \succ_{-k}\right]$ implying $\varphi\left[\succ_k^{*\prime}, \succ_i^* \succ_{-\{k,i\}}\right](k) = g'$. Yet, by submitting $\succ_k$ instead of $\succ_k^{*\prime}$, agent k improves his allocation, contradicting strategy-proofness.

Case 2. $\varphi\left[\succ_i^*, \succ_{-i}\right](k) \prec_k g$: By Claim 9, $\varphi\left[\succ_i^*, \succ_{-i}\right](i') \notin \{f, h\}$, and yet, when k submits $\succ_k^{*\prime}$ instead of $\succ_k$, he weakly improves his own allocation and strictly improves i''s allocation with respect to $\left[\succ_i^*, \succ_{-i}\right]$, a contradiction to coalitional strategy-proofness.

Thus, such distinct i and i' cannot exist. **QED**

Lemma 7 ***(Existence and uniqueness of a broker* for each b-house*)*** *Let $\sigma \in \mathcal{S} - \mathcal{M}$ and f be a b-house* at σ. Then there exists an agent $k \in I - I_\sigma$ who is the unique broker* of f at σ.*

Proof of Lemma 7. Let $\sigma \in \mathcal{S} - \mathcal{M}$ and f be a b-house* at σ. We start with the following preparatory:

Claim 1. Let $h, h' \in (H - H_\sigma) - \{f\}$ be such that $h \neq h'$, and let $\succ, \succ' \in \mathbf{P}[\sigma, f, h, h']$. Then $\varphi[\succ']^{-1}(h) = \varphi[\succ]^{-1}(h)$.

Proof of Claim 1. By Lemma 6, $\varphi[\succ']^{-1}(f) = \varphi[\succ]^{-1}(f)$. Let $i = \varphi[\succ]^{-1}(f)$. Also let $\succ^*$ and $\succ'^*$be monotonic extensions of $\succ$ and $\succ'$ respectively such that i ranks f first, all agents in $I - I_\sigma$ rank f below all houses in $(H - H_\sigma) - \{f\}$, and the relative ranking of all other houses at $\succ^*$, $\succ$ and $\succ'^*$, $\succ'$ are respectively the same. By Maskin monotonicity, $\varphi[\succ^{*\prime}] = \varphi[\succ']$ and $\varphi[\succ^*] = \varphi[\succ]$. Also $\succ^*, \succ'^* \in \mathbf{P}[\sigma \cup \{(i, f)\}, h, h']$. Thus, by Lemma 6, $\varphi[\succ^*]^{-1}(h) = \varphi[\succ^{*\prime}]^{-1}(h)$. Hence, $\varphi[\succ']^{-1}(h) = \varphi[\succ'^*]^{-1}(h) = \varphi[\succ^*]^{-1}(h) = \varphi[\succ]^{-1}(h)$. QED

Claim 2. Let $h, h' \in (H - H_\sigma) - \{f\}$ be such that $h \neq h'$ and let $\succ \in \mathbf{P}[\sigma, f, h, h']$ and $\succ' \in \mathbf{P}[\sigma, f, h']$ such that $\varphi[\succ']^{-1}(f) \neq \varphi[\succ]^{-1}(f)$. Then $\varphi[\succ']^{-1}(h') = \varphi[\succ]^{-1}(h)$.

Proof of Claim 2. Let $k' = \varphi[\succ']^{-1}(h')$ and $\succ^* \in \mathbf{P}[\sigma, f, h', h]$ be such that the only difference between $\succ^*$ and $\succ$ is the relative ranking of house h'. Since by Claim 1 $\varphi[\succ^*]^{-1}(h') = \varphi[\succ']^{-1}(h') = k'$

and since we push down house h' in everybody's preferences except k' at $[\succ^*_{k'}, \succ_{-k'}]$, by Maskin monotonicity $\varphi[\succ^*_{k'}, \succ_{-k'}] = \varphi[\succ^*]$. In particular, $\varphi[\succ^*_{k'}, \succ_{-k'}](k') = h'$. By strategy-proofness for k', we have $\varphi[\succ](k') \in \{h, h'\}$. On the other hand, by Lemma 6, $\varphi[\succ^*]^{-1}(f) = \varphi[\succ']^{-1}(f)$. Since $\varphi[\succ^*_{k'}, \succ_{-k'}] = \varphi[\succ^*]$, we have $\varphi[\succ^*_{k'}, \succ_{-k'}]^{-1}(f) = \varphi[\succ']^{-1}(f)$.

We also have, $\varphi[\succ]^{-1}(f) \neq \varphi[\succ']^{-1}(f) = \varphi[\succ^*_{k'}, \succ_{-k'}]^{-1}(f)$. Thus, by non-bossiness, agent k' should change his own allocation between the two profiles $\succ$ and $[\succ^*_{k'}, \succ_{-k'}]$, implying that $\varphi[\succ](k') = h$. QED

Claim 3. Let $h, h' \in (H - H_\sigma) - \{f\}$ be such that $h \neq h'$, $\succ \in \mathbf{P}[\sigma, f, h]$, and $\succ' \in \mathbf{P}[\sigma, f, h', h]$. Then, $\varphi[\succ]^{-1}(h) = \varphi[\succ']^{-1}(h')$.

Proof of Claim 3. If $\varphi[\succ]^{-1}(f) \neq \varphi[\succ']^{-1}(f)$, then we are done by Claim 2. Therefore, assume that $\varphi[\succ]^{-1}(f) = \varphi[\succ']^{-1}(f)$. Because f is a b-house* at σ, there exists some $h'' \in (H - H_\sigma) - \{f\}$ such that for some $\succ'' \in \mathbf{P}[\sigma, f, h'']$,

$$\varphi[\succ'']^{-1}(f) \neq \varphi[\succ]^{-1}(f) = \varphi[\succ']^{-1}(f).$$

By Lemma 6, $h'' \neq h$. By the same lemma, without loss of generality, we further assume that $\succ'' \in \mathbf{P}[\sigma, f, h'', h]$.

By Claim 2, $\varphi[\succ'']^{-1}(h'') = \varphi[\succ]^{-1}(h)$ and $\varphi[\succ'']^{-1}(h'') = \varphi[\succ']^{-1}(h')$, implying that $\varphi[\succ]^{-1}(h) = \varphi[\succ']^{-1}(h')$. QED

Claim 4. Let $h \in (H - H_\sigma) - \{f\}$ and $\succ, \succ' \in \mathbf{P}[\sigma, f, h]$. Then, $\varphi[\succ]^{-1}(h) = \varphi[\succ']^{-1}(h)$.

Proof of Claim 4. By Lemma 6, $\varphi[\succ]^{-1}(f) = \varphi[\succ']^{-1}(f)$. Because f is a b-house* at σ, there exists some $h'' \in (H - H_\sigma) - \{f\}$ such that for some $\succ'' \in \mathbf{P}[\sigma, f, h'']$,

$$\varphi[\succ'']^{-1}(f) \neq \varphi[\succ]^{-1}(f) = \varphi[\succ']^{-1}(f).$$

By Lemma 6, $h'' \neq h$. Let $\succ^* \in \mathbf{P}[\sigma, f, h'', h]$. By Claim 3, $\varphi[\succ^*]^{-1}(h'') = \varphi[\succ]^{-1}(h)$ and $\varphi[\succ^*]^{-1}(h'') = \varphi[\succ']^{-1}(h)$, implying that $\varphi[\succ]^{-1}(h) = \varphi[\succ']^{-1}(h)$. QED

We complete the proof of the lemma as follows: Let h and $h' \in H - H_\sigma - \{f\}$, $\succ \in \mathbf{P}[\sigma, f, h]$, $\succ' \in \mathbf{P}[\sigma, f, h']$. Two cases are needed:

Case 1. $h = h'$: Then $\varphi[\succ']^{-1}(h) = \varphi[\succ]^{-1}(h)$ by Claim 4.

Case 2. $h \neq h'$: Then let $\succ^* \in \mathbf{P}[\sigma, f, h, h']$; by Claim 3 $\varphi[\succ']^{-1}(h) = \varphi[\succ^*]^{-1}(h')$ and by Claim 4 $\varphi[\succ^*]^{-1}(h) = \varphi[\succ]^{-1}(h)$, implying that $\varphi[\succ]^{-1}(h) = \varphi[\succ']^{-1}(h')$.

Thus, the agent $\varphi[\succ]^{-1}(h)$ is the unique broker* of f at σ. **QED**

Lemma 8 ***(Ownership* is equivalent to strong ownership*)*** *Let $\sigma \in \mathcal{S} - \mathcal{M}$. An agent $i \in I - I_\sigma$ owns* any house $h \in H - H_\sigma$ at σ if and only if i strongly owns* h at σ.*

Proof of Lemma 8. Strong ownership* implying ownership* is straightforward from the definitions. We prove the other direction next. Let us start with two preparatory claims:

Claim 1. Let $\sigma \in \mathcal{S} - \mathcal{M}$, houses g and $h \in H - H_\sigma$ be such that $g \neq h$, and agent $i \in I - I_\sigma$ be such that i owns* h at σ. Then $\varphi[\succ_i^*, \succ_{-i}](i) = g$ for all $\succ_i^* \in \langle g, ...\rangle$ and all $\succ_{-i} \in \mathbf{P}_{-i}[\sigma, h]$.

Proof of Claim 1. Let $\succ_{-i} \in \mathbf{P}_{-i}[\sigma, h]$. Take any $\succ_i \in \langle h, g, ...\rangle$. If $\varphi[\succ](i) = h$, then Pareto efficiency and strategy-proofness imply that $\varphi[\succ_i^*, \succ_{-i}](i) = g$ for all $\succ_i^* \in \langle g, h, ...\rangle$, and furthermore, by strategy-proofness, for all $\succ_i^* \in \langle g, ...\rangle$. It remains to consider the case $\varphi[\succ](i) \neq h$.

Take $\succ' \in \mathbf{P}[\sigma, h, g]$ such that $\succ'$ and $\succ$ coincide other than unmatched agents' ranking of house g. Since i is the owner* of h at σ, we have $\varphi[\succ'](i) = h$. Two cases are possible: $\varphi[\succ](i) = g$ and $\varphi[\succ](i) \neq g$. If $\varphi[\succ]^{-1}(i) = g$, then by strategy-proofness, $\varphi[\succ_i^*, \succ_{-i}](i) = g$ and we are done. Thus, in the remainder assume that there exists some agent $k = \varphi[\succ]^{-1}(g) \neq i$. By Maskin monotonicity, $\varphi\left[\succ'_{\{i,k\}}, \succ_{-\{i,k\}}\right](i) = h$ and $\varphi\left[\succ'_{\{i,k\}}, \succ_{-\{i,k\}}\right](k) = g$.

Let $\succ_i^* \in \langle g, h, ...\rangle$. By strategy-proofness, agent i gets at least h at $\left[\succ_i^*, \succ'_k, \succ_{-\{i,k\}}\right]$; and by Pareto-efficiency, agent i gets g. Also recall that $\varphi[\succ](i) \prec_i g$ and $\varphi[\succ](k) = g$. Thus, $\varphi\left[\succ_i^*, \succ'_k, \succ_{-\{i,k\}}\right](k) \neq h$ because otherwise agents i and k could jointly improve upon their $\varphi[\succ]$ allocation by submitting $[\succ_i^*, \succ'_k]$ at $\succ$, contradicting coalitional strategy-proofness. Thus, $g \succ'_k \varphi\left[\succ_i^*, \succ'_k, \succ_{-\{i,k\}}\right](k)$, and furthermore, Maskin monotonicity implies $\varphi\left[\succ_i^*, \succ'_k, \succ_{-\{i,k\}}\right] = \varphi[\succ_i^*, \succ_{-i}]$. In particular, $\varphi[\succ_i^*, \succ_{-i}](i) = g$. QED

Claim 2. Let $\sigma \in \mathcal{S} - \mathcal{M}$, houses g and $h \in H - H_\sigma$ be such that $g \neq h$, $\succ \in \mathbf{P}[\sigma, h]$, and agent $i \in I - I_\sigma$ own* house h at σ. If there is some $\succ' \in \mathbf{P}[\sigma, h, g]$ such that $\succ_k \in \langle h, g, ...\rangle$ for $k = \varphi[\succ']^{-1}(g)$, then $\varphi[\succ](i) = h$.

Proof of Claim 2. By way of contradiction, assume that i is the owner* of h at σ, that $\succ' \in \mathbf{P}[\sigma, h, g]$, and that $k = \varphi[\succ']^{-1}(g)$, but there is some $\succ \in \mathbf{P}[\sigma, h]$ such that $\succ_k \in \langle h, g, ...\rangle$ and $\varphi[\succ]^{-1}(h) \neq i$. By strategy-proofness, we can choose $\succ_i \in \langle h, g, ...\rangle$. Furthermore, we can choose $\succ$ such that $\succ$ and $\succ'$ differ only in preferences of a single agent $j \in I - I_\sigma$ and in how house g is ranked by the agents.

Let $\succ^* \in \mathbf{P}[\sigma, h]$ be the unique profile, such that $\succ^*$ and $\succ$ differ only in the preferences of agent j, and $\succ^*$ and $\succ'$ differ only in how house g is ranked by the agents. Notice that $j \neq k$ as otherwise Maskin monotonicity would imply that i gets h at $\succ$. Thus, $\succ^*_k \in \langle h, g, ...\rangle$, and Maskin monotonicity implies that $\varphi[\succ^*](i) = h$.

Let h' be the house that j gets at $\succ$ and let $\succ''$ be the unique profile in $\mathbf{P}[\sigma, h, g]$ such that $\succ''$ and $\succ$ differ only in how house g is ranked by agents. By Maskin monotonicity, we may assume that $\succ''_j \in \langle h, g, h', ...\rangle$.

By Claim 1 and strategy-proofness, $\varphi\left[\succ''_j, \succ_{-j}\right](i)$ equals either h or g. At the same time strategy-proofness implies that $\varphi\left[\succ''_j, \succ_{-j}\right](j)$ equals either g or h'. In either case, agent j prefers the allocation of agent i at $\left[\succ''_j, \succ_{-j}\right]$. If $\varphi\left[\succ''_j, \succ_{-j}\right](i) = g$, this would be a contradiction with Lemma 3, as j could improve the allocation of i by switching from $\left[\succ''_j, \succ_{-j}\right]$ to $\left[\succ^*_j, \succ_{-j}\right] = \succ^*$. Hence, $\varphi\left[\succ''_j, \succ_{-j}\right](i) = h$, and by non-bossiness $\varphi\left[\succ''_j, \succ_{-j}\right](j) = g$. However, $k \neq j$ gets g at $\succ'$ and by strategy-proofness j cannot get it at $\left[\succ''_j, \succ'_{-j}\right]$. This is a contradiction because $\left[\succ''_j, \succ_{-j}\right] = \left[\succ''_j, \succ'_{-j}\right]$. QED

We are ready to finish the proof of the lemma. Fix $\sigma \in \mathcal{S} - \mathcal{M}$. We proceed by way of contradiction. Let i own* h at σ and $\succ \in \mathbf{P}[\sigma, h]$ be such that $\varphi[\succ]^{-1}(h) = j \neq i$. For all unmatched houses $g \neq h$ at σ, define $\succ^g$ to be the unique profile in $\mathbf{P}[\sigma, h, g]$ that differs from $\succ$ only in how agents rank g.

Take a house $g_1 \neq h$ unmatched at σ, and let k_1 be the agent that gets g_1 at $\succ^{g_1}$. By Claim 2, agent i gets h at any profile in $\mathbf{P}[\sigma, h]$ at which k_1 ranks g_1 second. Hence, by Maskin monotonicity i also gets h at any profile in $\mathbf{P}[\sigma, h]$ at which k_1 gets g_1.

Let $g_2 = \varphi[\succ](k_1)$ and let k_2 be the agent that gets g_2 at $\succ^{g_2}$. Because i does not get h at $\succ$, the previous paragraph yields $g_2 \neq g_1$ and $k_2 \neq k_1$. As in the previous paragraph, Claim 2 and Maskin monotonicity imply that i gets h at any profile in $\mathbf{P}[\sigma, h]$ at which k_2 gets g_2 or ranks g_2 second.

Furthermore, we will show that i gets h at any profile $\succ' \in \mathbf{P}[\sigma, h]$ at which k_2 ranks g_1 second. Indeed, suppose $\succ'_{k_2} \in \langle h, g_1, ...\rangle$ and i does not get h at $\succ'$. Let $\succ''_i \in \langle h, g_1, ...\rangle$. By Claim 1 and strategy-proofness, agent i gets g_1 at $\left[\succ''_i, \succ'_{-i}\right]$. By the previous paragraph and strategy-proofness, k_2 does not get h at $\left[\succ''_i, \succ'_{-i}\right]$, and thus k_2 envies i at $\left[\succ''_i, \succ'_{-i}\right]$. However, by the previous paragraph k_2 can improve the outcome of agent i, contrary to Lemma 3. Thus, i gets h at any profile in $\mathbf{P}[\sigma, h]$ at which k_2 ranks g_1 second.

Let g_3 be the house that k_2 gets at $\succ$ and let k_3 be the agent that gets g_3 at $\succ^{g_3}$. As above, we can show that i gets h at any profile in $\mathbf{P}[\sigma, h]$ at which k_3 ranks g_3 or g_2 or g_1 second.

Since the number of agents is finite, by repeating the procedure we arrive at an agent k_n who ranks one of the houses $g_1, ..., g_n$ second at $\succ$. That means that i gets h at $\succ$, a contradiction that concludes the proof. **QED**

Lemma 9 ***(Relationship between brokerage* and strong ownership*)****. Let $\sigma \in \mathcal{S} - \mathcal{M}$, agent k be a broker* of house f at σ, and $\succ'' \in \mathbf{P}^*[\sigma, f]$. Then agent $\varphi[\succ'']^{-1}(f)$ is the strong owner* of house $\varphi[\succ''](k)$ at σ.*

Proof of Lemma 9. Let $\succ'' \in \mathbf{P}^*[\sigma, f]$ and $h = \varphi[\succ''](k)$. Because k is a broker* at σ, Lemma 7 implies that house h is agent k's second choice. Since $\succ'' \in \mathbf{P}^*[\sigma, f]$, house h is the second choice of all agents in $I - I_\sigma$ at $\succ''$, and thus,

$$\succ'' \in \mathbf{P}[\sigma, f, h].$$

There exists an agent $i \in (I - I_\sigma) - \{k\}$ such that $\varphi[\succ'']^{-1}(f) = i$. By Lemma 6, for all $\succ \in \mathbf{P}[\sigma, f, h]$, agent i gets f at $\succ$. We are to show that i is the strong owner* of h at σ.

Claim 1. If $\succ \in \mathbf{P}[\sigma, f, h]$, then $\varphi[\succ](i) = f$ and $\varphi[\succ](k) = h$.

Proof of Claim 1. The first claim follows from Lemma 6, and the second from Lemma 7. QED

Claim 2. $\varphi[\succ](i) = f$ and $\varphi[\succ](k) = h$.

*Proof of Claim 2.*Let preference profile $\succ$ be such that $\succ_{i'} = \succ''_{i'}$ for all $i' \in \{k, i\} \cup I_\sigma$ and all houses in $H - H_\sigma$ are ranked above the houses in H_σ by $i' \in I - I_\sigma$. By Claim 1 and Maskin monotonicity, $\varphi[\succ](i) = f$ and $\varphi[\succ](k) = h$. QED.

Claim 3. $\varphi[\succ^*_i, \succ_{-i}](i) = h$.

Proof of Claim 3. Let $\succ_i^* \in \langle h, f, ...\rangle$. By strategy-proofness of φ, since $\varphi[\succ](i) = f$, agent i gets at least f at $[\succ_i^*, \succ_{-i}]$, and since all other agents in $I - I_\sigma$ prefer f over h, Pareto efficiency of φ implies that $\varphi[\succ_i^*, \succ_{-i}](i) = h$.

Claim 4. $\varphi[\succ_k^*, \succ_{-k}] = \varphi[\succ]$.

Proof of Claim 4. Let $\succ_k^* \in \langle h, f, ...\rangle$. Since $\varphi[\succ](k) = h$, profile $[\succ_k^*, \succ_{-k}]$ is a monotonic extension of $\succ$ and by Maskin monotonicity of φ, we have $\varphi[\succ_k^*, \succ_{-k}] = \varphi[\succ]$.

Claim 5. $\varphi\left[\succ_{\{i,k\}}^*, \succ_{-\{i,k\}}\right](i) = h$.

Proof of Claim 5. By Claim 4, $\varphi[\succ_k^*, \succ_{-k}](i) = \varphi[\succ](i) = f$, and, by strategy-proofness of φ, i gets at least f at $\left[\succ_{\{i,k\}}^*, \succ_{-\{i,k\}}\right]$. Thus, if i does not get h at $\left[\succ_{\{i,k\}}^*, \succ_{-\{i,k\}}\right]$ then one of the following two cases would have to obtain.

Case 1. An agent $j \notin \{i,k\}$ gets h at $\left[\succ_{\{i,k\}}^*, \succ_{-\{i,k\}}\right]$: Then i gets f, and k gets some house worse than f. But then jointly i and k can report $\succ_{\{i,k\}}$ instead of $\succ_{\{i,k\}}^*$ and they would jointly improve at $\succ_{\{i,k\}}^*$, i.e., $\varphi[\succ](i) = f = \varphi\left[\succ_{i,k}^*, \succ_{-i,k}\right](i)$ and $\varphi[\succ](k) = h \succ_k^* \varphi\left[\succ_{i,k}^*, \succ_{-i,k}\right](k)$, contradicting φ is coalitionally strategy-proof.

Case 2. Agent k gets h at $\left[\succ_{i,k}^*, \succ_{-i,k}\right]$: By strategy-proofness of φ, agent k should at least get h at $[\succ_i^*, \succ_{-i}]$. But we know by Step 2 that $\varphi[\succ_i^*, \succ_{-i}](i) = h$, thus we should have $\varphi[\succ_i^*, \succ_{-i}](k) = f$. Then by Maskin monotonicity of φ, we have $\varphi\left[\succ_{i,k}^*, \succ_{-i,k}\right](i) = \varphi[\succ_i^*, \succ_{-i}](i) = h$ where the last equality follows by Step 2. A contradiction that proves the claim. QED

Claim 6. If $\varphi\left[\succ_{\{i,k\}}^*, \succ_{-\{i,k\}}\right](i) = h$, then $\varphi\left[\succ_{\{i,k\}}^*, \succ_{-\{i,k\}}\right](k) \neq f$.

Proof of Claim 6. For an indirect argument, suppose that $\varphi\left[\succ_{\{i,k\}}^*, \succ_{-\{i,k\}}\right](i) = h$ and $\varphi\left[\succ_{\{i,k\}}^*, \succ_{-\{i,k\}}\right](k) = f$. Then, $\varphi[\succ_i^*, \succ_{-i}](k) = f$ by strategy-proofness of φ. Since f is a σ-b-house*, there exist some house $g \notin \{f, h\}$ and some preference profile $\succ' \in \mathbf{P}[\sigma, f, g]$ such that $\varphi[\succ']^{-1}(f) = j$ for some agent $j \notin \{i,k\}$. By Lemma 6, we may assume that each agent $i' \in I - I_\sigma$ ranks houses other than g and h in the same way at $\succ_{i'}'$ and $\succ_{i'}$ and that $\succ_{i'}' \in \langle f, g, h, ...\rangle$. Since k is the σ-broker* of f, we have $\varphi[\succ'](k) = g$. By Maskin monotonicity,

$$\varphi[\succ'] = \varphi\left[\succ_{\{i,k\}}', \succ_{-\{i,k\}}\right].$$

Now i gets a house weakly worse than h at $\left[\succ'_{\{i,k\}}, \succ_{-\{i,k\}}\right]$. However, if i and k manipulated and submitted $\succ^*_{\{i,k\}}$ instead of $\succ'_{\{i,k\}}$, they would get h and f respectively at $\left[\succ^*_{\{i,k\}}, \succ_{-\{i,k\}}\right]$. Both agents weakly improve, while k strictly improves. This contradicts the fact that φ is coalitionally strategy-proof. QED

Now, Claims 5 and 6 imply that $\varphi\left[\succ^*_{\{i,k\}}, \succ_{-\{i,k\}}\right](i) = h$ and $\varphi\left[\succ^*_{\{i,k\}}, \succ_{-\{i,k\}}\right](k) \neq f$. By Maskin monotonicity, we can drop the ranking of f in $\succ^*_i$ and $\succ^*_k$, and yet, the outcome of φ will not change. Recall that $\succ_{-\{i,k\}}$ was an arbitrary profile in which all houses in $H - H_\sigma$ are ranked above the houses in H_σ by $i' \in I - I_\sigma - \{i,k\}$. Thus, i gets h at all profiles of $\mathbf{P}[\sigma, h]$. **QED**

We state the following corollary to Lemmata 8 and 9:

Corollary 3 ***(Brokerage* and strong brokerage* are equivalent)*** *Let $\sigma \in \mathcal{S} - \mathcal{M}$. Pair (k, f) is a b-pair* at σ if and only if it is a strong b-pair* at σ.*

Lemma 10 ***(Uniqueness of a b-house*).*** *Let $\sigma \in \mathcal{S} - \mathcal{M}$. If f is a b-house* at σ, then no other house is a b-house* at σ (and all other unmatched houses are strong o-houses*).*

Proof of Lemma 10. Let f be a b-house* at σ. By Lemma 7, there is a broker* of f at σ, let us denote him as k. Consider a house $h \in I - I_\sigma - \{f\}$. By Lemma 6, there is an agent i who gets f at all profiles in $\mathbf{P}[\sigma, f, h]$. By Lemma 9, i is the strong owner* of h. Thus h is not a b-house* at σ. **QED**

Lemma 11 ***(Broker* does not own*).*** *Let $\sigma \in \mathcal{S} - \mathcal{M}$. If agent k is the broker* of house f at σ, then he cannot own* any houses at σ.*

Proof of Lemma 11. Suppose that k owns* a house $h \neq f$ at σ. By Lemma 6, there exists some agent $i \neq k$ who gets f at all profiles in $\mathbf{P}[\sigma, f, h]$. By Lemma 9, (i, h) is a strong o-pair* at σ. By definition, i gets h at all $\succ \in \mathbf{P}^*[\sigma, h]$, contradicting that k owns* h. **QED**

B.2 "Intertemporal" Lemmata (On Persistence of Ownership* and Brokerage*)

Lemma 12 ***(Persistence of ownership*).*** *Let (i,h) be an (strong) o-pair* at some $\sigma \in \mathcal{S}-\mathcal{M}$. If $\sigma' \supset \sigma$, and i and h are unmatched at σ', then (i,h) is a (strong) o-pair* at σ'.*

Proof of Lemma 12. Recall that ownership* and strong ownership* are equivalent by Lemma 8. Imagine to the contrary that i gets h at all $\succ \in \mathbf{P}[\sigma,h]$, but there is some $\succ' \in \mathbf{P}[\sigma',h]$ such that some agent $j \in I - I_{\sigma'}$, such that $j \neq i$, gets h at $\succ'$. Take $\succ \in \mathbf{P}[\sigma,h]$ such that

- for each agent $k \notin I_{\sigma'} - I_\sigma$, $\succ_k = \succ'_k$, and
- each agent $k \in I_{\sigma'} - I_\sigma$ ranks $\sigma'(k)$ as his second choice (just behind h) in $\succ_k$.

Each $k \in I_{\sigma'} - I_\sigma$ is indifferent between $\succ'$ and $\succ$ because:

- at $\succ'$ agent k gets $\sigma'(k)$ by Lemma 5,
- at $\succ$ agent k gets $\sigma'(k)$ by Pareto efficiency of φ and the fact that $\varphi[\succ](i) = h$.

The only difference between the profiles $\succ'$ and $\succ$ are the preferences of the agents in $I_{\sigma'} - I_\sigma$. Thus, agents $I_{\sigma'} - I_\sigma$ are indifferent between $\succ$ to $\succ'$ while agent j is strictly better off at $\succ'$. This contradicts the fact that φ is coalitionally strategy-proof. **QED**

Lemma 13 ***(Limited persistence of brokerage*)*** *Let $\sigma, \sigma' \in \mathcal{S}-\mathcal{M}$ be such that $\sigma' \supset \sigma$. Suppose that agent k is the σ-(strong-)broker* of house f, agent i is the σ-(strong-)owner* of house h, and agent $i' \neq i$ is the σ-(strong-)owner* of house h'. If k, i, i', f, h, h' are unmatched at σ', then (k,f) is still the (strong) b-pair* at σ'.*

Proof of Lemma 13. First, notice that i gets f at all $\succ \in \mathbf{P}[\sigma,f,h]$ and i' gets f at all $\succ \in \mathbf{P}[\sigma,f,h']$, and k gets h and h', respectively by Lemma 9. Take $\succ^h \in \mathbf{P}[\sigma,f,h]$ and $\succ^{h'} \in \mathbf{P}[\sigma,f,h']$ such that each agent $j \in I_{\sigma'} - I_\sigma$ has $\sigma'(j)$ as his third choice and each agent $j \in I - I_{\sigma'}$ ranks each house unmatched at σ' above all houses matched at σ' at both preference profiles. Let profile $\succ'^h$ be obtained from $\succ^h$ by moving $\sigma'(j)$ for all $j \in I_{\sigma'} - I_\sigma$ up to be the first choice of j. Let $\succ'^{h'}$ be obtained analogously from $\succ^{h'}$. By Maskin monotonicity, $\varphi\left[\succ'^{h}\right]^{-1}(f) = i \neq i' = \varphi[\succ'^{h'}]^{-1}(f)$. Since $\succ'^{h}$ and $\succ'^{h'} \in \mathbf{P}^*[\sigma', f]$, house f is a b-house* at σ'.

For an indirect argument for the second part of the proof, suppose that k is not the broker* of f at σ'. Then, by Lemma 7 there exists some other agent $k' \neq k$ who brokers* f at σ'.

Let $\succ' \in \mathbf{P}[\sigma', f, h]$ be arbitrary and $\succ \in \mathbf{P}[\sigma, f, h]$ be such that each agent j in $I_{\sigma'} - I_{\sigma}$ lists $\sigma'(j)$ as his third choice at $\succ$, each agent in $I - I_{\sigma'}$ lists houses in $H_{\sigma'}$ lower than houses in $H_{\sigma'} - H_{\sigma}$ at $\succ$, and rest of the relative rankings of the houses are the same between $\succ$ and $\succ'$. Since (k, f) is a σ-b-pair* at σ and (i, h) is an o-pair* at σ, by Lemma 9 $\varphi[\succ](k) = g$ and $\varphi[\succ](i) = f$. Then, by Pareto efficiency, $\varphi[\succ'](j) = \sigma'(j)$ for all $j \in I_{\sigma'} - I_{\sigma}$, and thus, by Maskin monotonicity, $\varphi[\succ'] = \varphi[\succ]$. Now, $\varphi[\succ'](k) = h$, however, this contradicts the fact that agent $k' \neq k$ brokers* f at σ' and thus, $\varphi[\succ'](k') = h$. Therefore, (k, f) is the b-pair* at σ', as well. **QED**

Lemma 14 ***(Broker*-to-heir* transition)*** *Let $\sigma \in \mathcal{S} - \mathcal{M}$, $k, j, i \in I - I_{\sigma}$, and $f, g, h \in H - H_{\sigma}$ be such that $k \neq j$ (strongly) brokers* $f \neq g$ at σ, i is the only (strong) owner* who is unmatched at both σ and $\sigma' = \sigma \cup \{(j, g)\}$ and (strongly) owns* h both at σ and σ'. Further suppose that (k, f) is no longer a (strong) b-pair* at σ'. Then i (strongly) owns* f at σ' and k (strongly) owns* h at $\sigma' \cup \{(i, f)\}$.*

Proof of Lemma 14. By Lemmata 8 and 9 and Maskin monotonicity, for all profiles $\succ \in \mathbf{P}[\sigma]$ such that $\succ_i \in \langle f, ...\rangle$, $\succ_k \in \langle f, h, ...\rangle$ at σ, we have $\varphi[\succ](i) = f$ and $\varphi[\succ](k) = h$. Since $\mathbf{P}[\sigma'] \subsetneq \mathbf{P}[\sigma]$, either (i, f) is an (strong) o-pair* at σ' or (i, f) is a (strong) b-pair* at σ'. Since the latter is not true, then (i, f) is an (strong) o-pair* at σ'. Let $\succ' \in \mathbf{P}[\sigma' \cup \{(i, f)\}, h]$. Fix a profile $\succ$ as described above with the further restriction that $\succ \in \mathbf{P}[\sigma', f]$ and relative ranking of all houses except h and f coincides with that of $\succ'$. Thus, $\succ'$ is a monotonic extension of $\succ$, implying that $\varphi[\succ'] = \varphi[\succ]$, and in particular, $\varphi[\succ'](k) = \varphi[\succ](k) = h$. Thus, (k, h) is an (strong) o-pair* at $\sigma' \cup \{(i, f)\}$. **QED**

Lemma 15 *Let $\sigma \in \mathcal{S} - \mathcal{M}$, such that there exists a unique agent i unmatched at σ. Then i σ-(strongly)-owns* all unmatched houses at $\sigma \in I - I_{\sigma}$.*

Proof of Lemma 15. Let $\succ \in \mathbf{P}[\sigma, h]$ for $h \in H - H_{\sigma}$. By Pareto efficiency of φ, $\varphi[\succ](i) = h$, implying that i (strongly) owns* h at σ. **QED**

B.3 Proof of Theorem 4

Proof of Theorem 4. Let φ be a coalitionally strategy-proof and Pareto-efficient mechanism. We will construct a compatible control rights structure (c, b). Fix $\sigma \in \mathcal{S} - \mathcal{M}$. For all $h \in H - H_\sigma$, two cases are possible:

Case 1. $\varphi[\succ]^{-1}(h)$ is constant for all $\succ \in \mathbf{P}^*[\sigma, h]$: Then $\left(\varphi[\succ]^{-1}(h), h\right)$ is an (strong) o-pair* at σ. We set

$$c_h(\sigma) = \varphi[\succ]^{-1}(h).$$

Case 2. There exist $\succ, \succ' \in \mathbf{P}^*[\sigma, h]$ such that $\varphi[\succ]^{-1}(h) \neq \varphi[\succ']^{-1}(h)$: Then there exists $k \in I - I_\sigma$, such that (k, h) is a (strong) b-pair* (that is, for all $\succ^* \in \mathbf{P}^*[\sigma, h]$, $\varphi[\succ^*](k)$ is the second choice of agent k at $\succ^*_k$). In this case, we set

$$c_h(\sigma) = k \text{ and}$$
$$b(\sigma) = h.$$

Additionally, if Case 2 does not hold for any house $h \in H - H_\sigma$, then we set

$$b(\sigma) = \emptyset.$$

First, we state the following claim that links control* rights structure induced by φ with control rights structure (c, b):

Claim 1. Pair (c, b) is a well-defined compatible control-rights structure. Moreover, o-pairs* and b-pair* induced by φ coincide with o-pairs and b-pairs induced by (c, b) at all $\sigma \in \mathcal{S} - \mathcal{M}$.

Proof of Claim 1. Let $\sigma \in \mathcal{S} - \mathcal{M}$.By Lemma 8 and Corollary 3, strong o-pairs* and o-pairs*, strong b-pair* and b-pair* (if it exists) are the same at σ, respectively. Thus, by construction, all o-pairs coincide with o-pair* and the b-pair coincide with the b-pair*, if it exists, at σ.

At σ, there exists at most one b-house (by Lemma 10) which is brokered by a unique broker (by 7). Moreover if there exists a unique agent unmatched at σ, then he owns all remaining unmatched houses (by Lemma 15). Thus, (c, b) is a well-defined control rights structure.

Lemma 12 shows ownership persists (C1). Lemma 11 shows that brokers do not own (C2). Suppose (k, f) is a b-pair at σ and that $\sigma \cup \{(j, g)\} \supsetneq \sigma$ is such that $j \neq k$ and $g \neq f$ are unmatched at σ. There are three cases about the σ-owners:

Case 1. There are at least two σ-owners unmatched at $\sigma \cup \{(j, g)\}$: Lemma 13 shows that (k, f) is a b-pair at $\sigma \cup \{(j, g)\}$ (this is the limited persistence of brokerage).

Case 2. One σ-owner i is unmatched at $\sigma \cup \{(j, g)\}$: Either (k, f) is still a b-pair at $\sigma \cup \{(j, g)\}$ or it is not a b-pair, and by Lemma 14, agent i owns at $\sigma \cup \{(j, g)\}$ house f and agent k owns at $\sigma \cup \{(j, g), (i, f)\}$ all σ-owned houses of i (this is the broker-to-heir transition).

Case 3. No σ-owners are unmatched at $\sigma \cup \{(j, g)\}$: Either (k, f) is still a b-pair at $\sigma \cup \{(j, g)\}$ or it is no longer a b-pair (the former is the persistence of and the latter is the direct exit from brokerage).

We showed that (c, b) also satisfies C3, and hence, it is compatible. QED

To prove that $\varphi = \psi^{c,b}$, fix $\succ \in \mathbf{P}$. We will show that $\varphi[\succ] = \psi^{c,b}[\succ]$. Let I^r be the set of agents removed in round r of $\psi^{c,b}$. For each agent $i \in I^r$, there is a unique house that points to him and is removed in the same cycle as i; let us denote this house h_i. Let us construct the following preference profile $\succ^*$ by modifying $\succ$.

- If $\psi^{c,b}[\succ](i) = h_i$, then $\succ_i^* = \succ_i$.
- If $\psi^{c,b}[\succ](i) \neq h_i$ and if no b-house was removed in the same cycle as i or the b-house was assigned to i, then we construct $\succ_i^*$ from $\succ_i$ by moving h_i just after $\psi^{c,b}[\succ](i)$ (we do not change the ranking of other houses).
- If i is removed as owner and a b-house $f^r \neq \psi^{c,b}[\succ](i)$ was removed in the same cycle as i, then we construct $\succ_i^*$ from $\succ_i$ by moving f^r just after $\psi^{c,b}[\succ](i)$ and moving h_i just after f^r.
- If a broker k^r was removed in the cycle

$$h_{i^1} \to i^1 \to h_{i^2} \to i^2 \to ...h_{i^n} \to i^n \to f^r \to k^r \to h_{i^1},$$

then we construct $\succ_{k^r}^*$ from $\succ_{k^r}$ by moving h_{i^n} just below h_{i^1}.

Observe that $\psi^{c,b}[\succ^*] = \psi^{c,b}[\succ]$. Moreover, since

$$\left\{ h \in H : h \succeq_i \underbrace{\psi^{c,b}[\succ](i)}_{=\psi^{c,b}[\succ^*](i)} \right\} = \left\{ h \in H : h \succeq_i^* \underbrace{\psi^{c,b}[\succ](i)}_{=\psi^{c,b}[\succ^*](i)} \right\} \qquad \forall i \in I, \tag{2}$$

$\succ^*$ is a monotonic extension of $\succ$ at $\psi^{c,b}$ and $\succ$ is a monotonic extension of $\succ^*$ at $\psi^{c,b}$.

We will next prove that

$$\varphi[\succ^*](i) = \psi^{c,b}[\succ^*](i) \qquad \forall i \in \cup_{s \le r} I^s = I_{\sigma^r}, \qquad \forall r = 0, 1, 2, ... \tag{3}$$

by induction over r. The claim is trivially true for $r = 0$. Fix round $r \geq 1$ and let σ^{r-1} be the matching fixed before round r (in particular, $\sigma^0 = \varnothing$). For the inductive step, assume that

$$\varphi[\succ^*](i) = \psi^{c,b}[\succ^*](i) \qquad \forall i \in \cup_{s \le r-1} I^s = I_{\sigma^{r-1}} \tag{4}$$

for . We will prove that the same expression holds for agents in I^r using the following three claims.

Claim 2. $\varphi[\succ^*](i) \succeq_i^* h_i$ for all owners $i \in I^r$.

Proof of Claim 2. Let $\succ' \in \mathbf{P}[\sigma^{r-1}, h_i]$ be a preference profile such that the relative ranking of all houses in $H - H_{\sigma^{r-1}} - \{h_i\}$ in $\succ'_j$ is the same as in $\succ^*_j$ for all $j \in (I - I_{\sigma^{r-1}}) - \{i\}$, and let $\succ'' \in \mathbf{P}[\sigma^{r-1}]$ be a preference profile such that the relative ranking of all houses in $H - H_{\sigma^{r-1}}$ in $\succ''_j$ is the same as in $\succ^*_j$ for all $j \in (I - I_{\sigma^{r-1}}) - \{i\}$.

If $i' \in I_{\sigma^{r-1}}$ then

$$\varphi[\succ'](i') = \varphi[\succ''](i') = \sigma^{r-1}(i') = \psi^{c,b}[\succ^*](i') = \varphi[\succ^*](i'),$$

by construction of $\mathbf{P}[\sigma^{r-1}, h_i], \mathbf{P}[\sigma^{r-1}]$, and σ^{r-1}, and by the inductive assumption. Since (i, h_i) is an o-pair at σ^{r-1}, it is an o-pair* by construction of (c, b). By equivalence of strong ownership* and ownership* (Lemma 8),

$$\varphi[\succ'](i) = h_i. \tag{5}$$

Thus, no agent $j \in (I - I_{\sigma^{r-1}}) - \{i\}$ gets a house in $\{h_i\} \cup H_{\sigma^{r-1}}$ at $\varphi[\succ']$.

By Maskin monotonicity,

$$\begin{aligned} \varphi[\succ^*] &= \varphi\left[\succ''_{(I - I_{\sigma^{r-1}}) - \{i\}}, \succ^*_{I_{\sigma^{r-1}} \cup \{i\}}\right] \\ &= \varphi\left[\succ''_{(I - I_{\sigma^{r-1}}) - \{i\}}, \succ'_{I_{\sigma^{r-1}}}, \succ^*_i\right], \end{aligned} \tag{6}$$

and

$$\varphi\left[\succ'\right]=\varphi\left[\succ''_{\left(I-I_{\sigma^{r-1}}\right)-\{i\}},\succ'_{I_{\sigma^{r-1}}\cup\{i\}}\right]. \tag{7}$$

By Equation 6, strategy-proofness of φ, and Equations 7 and 5, we have

$$\varphi\left[\succ^*\right](i)=\varphi\left[\succ''_{\left(I-I_{\sigma^{r-1}}\right)-\{i\}},\succ'_{I_{\sigma^{r-1}}},\succ^*_i\right](i)\ \succeq^*_i\ \varphi\left[\succ''_{\left(I-I_{\sigma^{r-1}}\right)-\{i\}},\succ'_{I_{\sigma^{r-1}}\cup\{i\}}\right](i)=\varphi\left[\succ'\right](i)=h_i.$$

QED

Claim 3. If $i\in I^r$ and no b-house was removed in the cycle of i, then $\varphi\left[\succ^*\right](i)=\psi^{c,b}\left[\succ^*\right](i)$.

Proof of Claim 3. The inductive assumption implies that all houses better than $\psi^{c,b}\left[\succ^*\right](i)$ are already given to other agents, hence

$$\psi^{c,b}\left[\succ^*\right](i)\succeq^*_i\varphi\left[\succ^*\right](i).$$

For an indirect argument, suppose $\varphi\left[\succ^*\right](i)\neq\psi^{c,b}\left[\succ^*\right](i)$. Then, Claim 2 and the construction of $\succ^*$ imply that

$$\varphi\left[\succ^*\right](i)=h_i.$$

Let

$$h_i\rightarrow i\rightarrow h_{i^2}\rightarrow i^2\rightarrow ...\rightarrow h_{i^n}\rightarrow i^n\rightarrow h_i$$

be the cycle in which i is removed under $\psi^{c,b}\left[\succ^*\right]$. From

$$\varphi\left[\succ^*\right](i)=h_i=\psi^{c,b}\left[\succ^*\right]\left(i^n\right),$$

we conclude that $\varphi\left[\succ^*\right]\left(i^n\right)\neq\psi^{c,b}\left[\succ^*\right]\left(i^n\right)$, and Claim 2 and the construction of $\succ^*$ imply that

$$\varphi\left[\succ^*\right]\left(i^n\right)=h_{i^n}=\psi^{c,b}\left[\succ^*\right]\left(i^{n-1}\right).$$

As we continue iteratively, we obtain that

$$\varphi\left[\succ^*\right](j)=h_j$$

for all $j\in\{i,i^2,...,i^n\}$. Hence, the matching obtained by assigning $\psi^{c,b}\left[\succ^*\right](j)$ to each agent $j\in\{i,i^2,...,i^n\}$ and $\varphi\left[\succ^*\right](j)$ to each agent $j\in I-\{i,i^2,...,i^n\}$ Pareto-dominates $\varphi\left[\succ^*\right]$ at $\succ^*$, contradicting that $\varphi\left[\succ^*\right]$ is Pareto-efficient. QED

Claim 4. If $i \in I^r$ and a b-house was removed in the cycle of i, then $\varphi[\succ^*](i) = \psi^{c,b}[\succ^*](i)$.

Proof of Claim 4. Let f be the σ^{r-1}-b-house and k be the σ^{r-1}-broker removed in the cycle

$$h_{i^1} \to i^1 \to h_{i^2} \to ... \to i^n \to f \to k \to h_{i^1}$$

Let

$$h_{i^{n+1}} \equiv f \qquad \text{and} \qquad i^0 \equiv k.$$

For all $i^\ell \in \{i^1, ..., i^n\}$, by the inductive assumption, all houses better than $h_{i^{\ell+1}}$ are already given to other agents, hence Claim 2 implies that

$$\varphi[\succ^*]\left(i^\ell\right) \in \{h_{i^{\ell+1}}, f, h_{i^\ell}\}, \tag{8}$$

where $h^*_{i^{\ell+1}} \succeq^*_{i^\ell} f \succ_{i^\ell} h_{i^\ell}$. We prove the claim in two steps:

- First, we show that

  $$\varphi[\succ^*](i^n) = f = \psi^{c,b}[\succ^*](i^n).$$

  Suppose on the contrary that $\varphi[\succ^*](i^n) \neq f$. Then, $\varphi[\succ^*](i^n) = h_{i^n}$ by Equation 8. By iteration of the same argument for $\ell = n-1, n-2, ..., 1$, we have

  $$\varphi[\succ^*]\left(i^\ell\right) \in \{f, h_{i^\ell}\}. \tag{9}$$

  Recall that by construction $f \succ^*_{i^\ell} h_{i^\ell}$. Let $\succ' \in \mathbf{P}[\sigma^{r-1}]$ be such that the relative ranking of all houses in $H - H_{\sigma^{r-1}}$ at $\succ'_j$ is the same as at $\succ^*_j$ for all $j \in I - \{k, i^1\}$ and $\succ'_k, \succ'_{i^1} \in \langle f, h_{i^1}, ...\rangle$. Then by Lemma 9, $\varphi[\succ'](i^1) = f$ and $\varphi[\succ'](k) = h_{i^1}$. Thus, by Maskin monotonicity, $\varphi[\succ'] = \varphi[\succ^*]$, implying that $\varphi[\succ^*](i^1) = f$ and $\varphi[\succ'](k) = h_{i^1}$. Moreover, Equation 9 implies that $\varphi[\succ^*]\left(i^\ell\right) = h_{i^\ell} \quad \forall \ell \in \{1, ..., n-1\}$. However, the matching that assigns each agent in $i^\ell \in \{i^1, ..., i^n\}$ the house $h_{i^{\ell+1}}$ and every other agent j the house $\varphi[\succ^*](j)$ Pareto-dominates $\varphi[\succ^*]$, contradicting Pareto-efficiency of φ.

- Next, we show that

  $$\varphi[\succ^*]\left(i^\ell\right) = h_{i^{\ell+1}} = \psi^{c,b}[\succ^*]\left(i^\ell\right) \qquad \forall\, \ell \in \{0, ..., n-1\}.$$

  On the contrary, suppose there exists some $\ell \in \{0, ..., n-1\}$ such that $\varphi[\succ^*]\left(i^\ell\right) \neq h_{i^{\ell+1}}$. Thus, by Equation 8 and the fact that $\varphi[\succ^*](i^n) = f$, we have $\varphi[\succ^*]\left(i^\ell\right) = h_{i^\ell}$. By iteration of this argument for all $m = \ell - 1, \ell - 2, ..., 1$, $\varphi[\succ^*](i^m) = h_{i^m}$. Thus, $\varphi[\succ^*](k) \neq h_{i^1}$. Let $\succ' \in \mathbf{P}[\sigma^{r-1}]$ be such that the relative ranking of all houses in $H - H_{\sigma^{r-1}}$ at $\succ'_j$ is the same as at $\succ^*_j$ for all $j \in I - \{k, i^n\}$, and $\succ'_k, \succ'_{i^n} \in \langle f, h_{i^n}, ...\rangle$. Then by Lemma 9, $\varphi[\succ'](i^n) = f$ and $\varphi[\succ'](k) = h_{i^n}$. By $\varphi[\succ^*](k) \neq h_{i^1}$ and then Maskin monotonicity, $\varphi[\succ^*] = \varphi[\succ']$, and in particular, $\varphi[\succ^*](k) = h_{i^n}$. Thus, by Equation 8, $\varphi[\succ'](i^{n-1}) = h_{i^{n-1}}$. By iteration of the same argument for all $\ell = n-2, n-3, ..., \ell+1$, $\varphi[\succ'](i^m) = h_{i^m}$. On the other hand, the matching which assigns each agent $i^\ell \in \{i^1, ..., i^{n-1}\}$ house $h_{i^{\ell+1}}$, agent k house h_{i^1} and all other agents their houses at $\varphi[\succ^*]$ Pareto-dominates $\varphi[\succ^*]$, contradicting Pareto efficiency of φ. QED

Let σ^r be the matching fixed after Round r. By the inductive assumption, and by Claims 3 and 4, $\varphi[\succ^*](i) = \psi^{c,b}[\succ^*](i)$ for all $i \in I_{\sigma^r}$. This completes the induction, and the proof of Statement in (3) (i.e., Equation 3).

The theorem follows from

$$\psi^{c,b}[\succ] = \psi^{c,b}[\succ^*], \qquad \psi^{c,b}[\succ^*] = \varphi[\succ^*], \qquad \text{and} \qquad \varphi[\succ^*] = \varphi[\succ].$$

The first of these observations is straightforward through the construction of $\succ^*$. The second one follows from Equation 3. The third one follows from Maskin monotonicity of φ, because $\psi^{c,b}[\succ^*] = \varphi[\succ^*]$ and Equation 2 together imply that

$$\{h \in H : h \succeq_i \varphi[\succ^*](i)\} = \{h \in H : h \succeq^*_i \varphi[\succ^*](i)\} \text{ for all } i \in I.$$

QED

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