

# 14.452 Economic Growth: Lecture 2: The Solow Growth Model

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# Review of the Discrete-Time Solow Model

- Per capita capital stock evolves according to

$$k(t+1) = sf(k(t)) + (1 - \delta)k(t). \quad (1)$$

- The steady-state value of the capital-labor ratio  $k^*$  is given by

$$\frac{f(k^*)}{k^*} = \frac{\delta}{s}. \quad (2)$$

- Consumption is given by

$$C(t) = (1 - s)Y(t) \quad (3)$$

- And factor prices are given by

$$\begin{aligned} R(t) &= f'(k(t)) > 0 \text{ and} \\ w(t) &= f(k(t)) - k(t)f'(k(t)) > 0. \end{aligned} \quad (4)$$

# Steady State Equilibrium

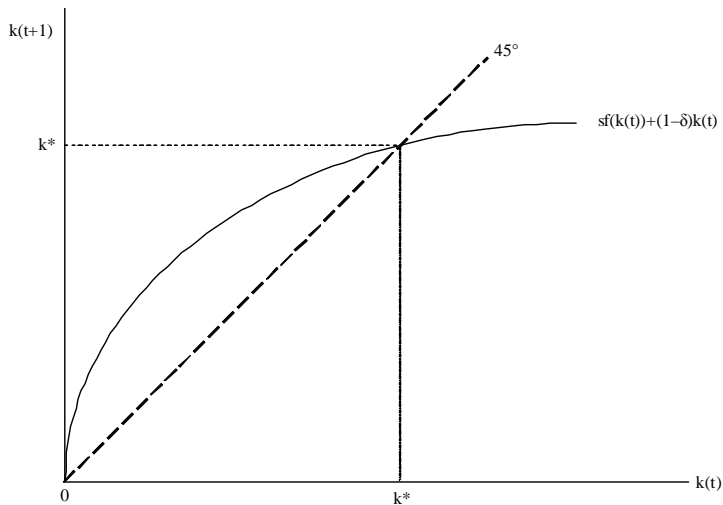


Figure: Steady-state capital-labor ratio in the Solow model.

# Transitional Dynamics

- *Equilibrium path*: not simply steady state, but entire path of capital stock, output, consumption and factor prices.
  - In engineering and physical sciences, equilibrium is point of rest of dynamical system, thus *the steady state equilibrium*.
  - In economics, non-steady-state behavior also governed by optimizing behavior of households and firms and market clearing.
- Need to study the “transitional dynamics” of the equilibrium difference equation (1) starting from an arbitrary initial capital-labor ratio  $k(0) > 0$ .
- Key question: whether economy will tend to steady state and how it will behave along the transition path.

# Transitional Dynamics: Review I

- Consider the nonlinear system of autonomous difference equations,

$$\mathbf{x}(t+1) = \mathbf{G}(\mathbf{x}(t)), \quad (5)$$

- $\mathbf{x}(t) \in \mathbb{R}^n$  and  $\mathbf{G} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ .
- Let  $\mathbf{x}^*$  be a fixed point of the mapping  $\mathbf{G}(\cdot)$ , i.e.,

$$\mathbf{x}^* = \mathbf{G}(\mathbf{x}^*).$$

- $\mathbf{x}^*$  is sometimes referred to as “an equilibrium point” of (5).
- We will refer to  $\mathbf{x}^*$  as a stationary point or a *steady state* of (5).

**Definition** A steady state  $\mathbf{x}^*$  is (locally) *asymptotically stable* if there exists an open set  $B(\mathbf{x}^*) \ni \mathbf{x}^*$  such that for any solution  $\{\mathbf{x}(t)\}_{t=0}^{\infty}$  to (5) with  $\mathbf{x}(0) \in B(\mathbf{x}^*)$ , we have  $\mathbf{x}(t) \rightarrow \mathbf{x}^*$ . Moreover,  $\mathbf{x}^*$  is *globally asymptotically stable* if for all  $\mathbf{x}(0) \in \mathbb{R}^n$ , for any solution  $\{\mathbf{x}(t)\}_{t=0}^{\infty}$ , we have  $\mathbf{x}(t) \rightarrow \mathbf{x}^*$ .

# Transitional Dynamics: Review II

## Simple Result About Stability

- Let  $x(t)$ ,  $a, b \in \mathbb{R}$ , then the unique steady state of the linear difference equation  $x(t+1) = ax(t) + b$  is globally asymptotically stable (in the sense that  $x(t) \rightarrow x^* = b/(1-a)$ ) if  $|a| < 1$ .
- Suppose that  $g: \mathbb{R} \rightarrow \mathbb{R}$  is differentiable at the steady state  $x^*$ , defined by  $g(x^*) = x^*$ . Then, the steady state of the nonlinear difference equation  $x(t+1) = g(x(t))$ ,  $x^*$ , is locally asymptotically stable if  $|g'(x^*)| < 1$ . Moreover, if  $|g'(x)| < 1$  for all  $x \in \mathbb{R}$ , then  $x^*$  is globally asymptotically stable.

# Transitional Dynamics in the Discrete Time Solow Model

**Proposition** Suppose that Assumptions 1 and 2 hold, then the steady-state equilibrium of the Solow growth model described by the difference equation (1) is globally asymptotically stable, and starting from any  $k(0) > 0$ ,  $k(t)$  monotonically converges to  $k^*$ .

# Proof of Proposition: Transitional Dynamics I

- Let  $g(k) \equiv sf(k) + (1 - \delta)k$ . First observe that  $g'(k)$  exists and is always strictly positive, i.e.,  $g'(k) > 0$  for all  $k$ .
- Next, from (1),

$$k(t+1) = g(k(t)), \quad (6)$$

with a unique steady state at  $k^*$ .

- From (2), the steady-state capital  $k^*$  satisfies  $\delta k^* = sf(k^*)$ , or

$$k^* = g(k^*). \quad (7)$$

- Recall that  $f(\cdot)$  is concave and differentiable from Assumption 1 and satisfies  $f(0) \geq 0$  from Assumption 2.

## Proof of Proposition: Transitional Dynamics II

- For any strictly concave differentiable function,

$$f(k) > f(0) + kf'(k) \geq kf'(k), \quad (8)$$

- The second inequality uses the fact that  $f(0) \geq 0$ .
- Since (8) implies that  $\delta = sf(k^*)/k^* > sf'(k^*)$ , we have  $g'(k^*) = sf'(k^*) + 1 - \delta < 1$ . Therefore,

$$g'(k^*) \in (0, 1).$$

- The Simple Result then establishes local asymptotic stability.

## Proof of Proposition: Transitional Dynamics III

- To prove global stability, note that for all  $k(t) \in (0, k^*)$ ,

$$\begin{aligned}k(t+1) - k^* &= g(k(t)) - g(k^*) \\ &= - \int_{k(t)}^{k^*} g'(k) dk, \\ &< 0\end{aligned}$$

- First line follows by subtracting (7) from (6), second line uses the fundamental theorem of calculus, and third line follows from the observation that  $g'(k) > 0$  for all  $k$ .

# Proof of Proposition: Transitional Dynamics IV

- Next, (1) also implies

$$\begin{aligned} \frac{k(t+1) - k(t)}{k(t)} &= s \frac{f(k(t))}{k(t)} - \delta \\ &> s \frac{f(k^*)}{k^*} - \delta \\ &= 0, \end{aligned}$$

- Second line uses the fact that  $f(k)/k$  is decreasing in  $k$  (from (8) above) and last line uses the definition of  $k^*$ .
- These two arguments together establish that for all  $k(t) \in (0, k^*)$ ,  $k(t+1) \in (k(t), k^*)$ .
- An identical argument implies that for all  $k(t) > k^*$ ,  $k(t+1) \in (k^*, k(t))$ .
- Therefore,  $\{k(t)\}_{t=0}^{\infty}$  monotonically converges to  $k^*$  and is globally stable.

# Transitional Dynamics in the Discrete Time Solow Model

## III

- Stability result can be seen diagrammatically in the Figure:
  - Starting from initial capital stock  $k(0) < k^*$ , economy grows towards  $k^*$ , *capital deepening* and growth of per capita income.
  - If economy were to start with  $k'(0) > k^*$ , reach the steady state by decumulating capital and contracting.

**Proposition** Suppose that Assumptions 1 and 2 hold, and  $k(0) < k^*$ , then  $\{w(t)\}_{t=0}^{\infty}$  is an increasing sequence and  $\{R(t)\}_{t=0}^{\infty}$  is a decreasing sequence. If  $k(0) > k^*$ , the opposite results apply.

- Thus far Solow growth model has a number of nice properties, but no growth, except when the economy starts with  $k(0) < k^*$ .

# Transitional Dynamics in Figure

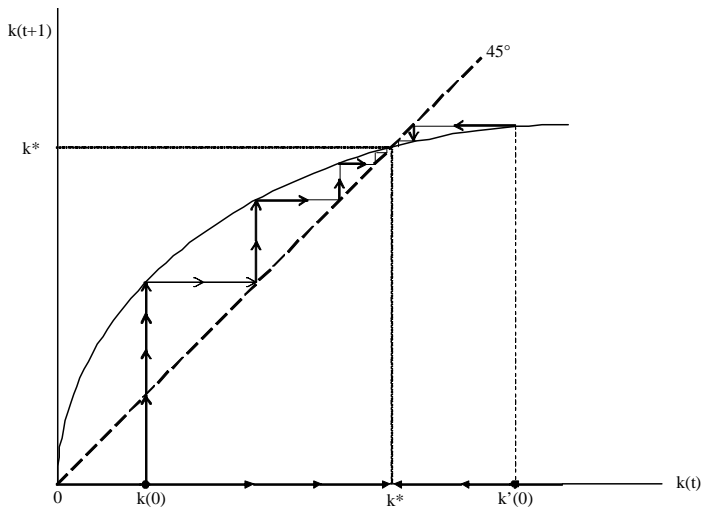


Figure: Transitional dynamics in the basic Solow model.

# From Difference to Differential Equations I

- Start with a simple difference equation

$$x(t+1) - x(t) = g(x(t)). \quad (9)$$

- Now consider the following approximation for any  $\Delta t \in [0, 1]$ ,

$$x(t + \Delta t) - x(t) \simeq \Delta t \cdot g(x(t)),$$

- When  $\Delta t = 0$ , this equation is just an identity. When  $\Delta t = 1$ , it gives (9).
- In-between it is a linear approximation, not too bad if  $g(x) \simeq g(x(t))$  for all  $x \in [x(t), x(t+1)]$

# From Difference to Differential Equations II

- Divide both sides of this equation by  $\Delta t$ , and take limits

$$\lim_{\Delta t \rightarrow 0} \frac{x(t + \Delta t) - x(t)}{\Delta t} = \dot{x}(t) \simeq g(x(t)), \quad (10)$$

where

$$\dot{x}(t) \equiv \frac{dx(t)}{dt}$$

- Equation (10) is a differential equation representing (9) for the case in which  $t$  and  $t + 1$  is “small”.

# The Fundamental Equation of the Solow Model in Continuous Time I

- Nothing has changed on the production side, so (4) still give the factor prices, now interpreted as instantaneous wage and rental rates.
- Savings are again

$$S(t) = sY(t),$$

- Consumption is given by (3) above.
- Introduce population growth,

$$L(t) = \exp(nt) L(0). \quad (11)$$

- Recall

$$k(t) \equiv \frac{K(t)}{L(t)},$$

# The Fundamental Equation of the Solow Model in Continuous Time II

- Implies

$$\begin{aligned}\frac{\dot{k}(t)}{k(t)} &= \frac{\dot{K}(t)}{K(t)} - \frac{\dot{L}(t)}{L(t)}, \\ &= \frac{\dot{K}(t)}{K(t)} - n.\end{aligned}$$

- From the limiting argument leading to equation (10),

$$\dot{K}(t) = sF[K(t), L(t), A(t)] - \delta K(t).$$

- Using the definition of  $k(t)$  and the constant returns to scale properties of the production function,

$$\frac{\dot{k}(t)}{k(t)} = s \frac{f(k(t))}{k(t)} - (n + \delta), \quad (12)$$

# The Fundamental Equation of the Solow Model in Continuous Time III

**Definition** In the basic Solow model in continuous time with population growth at the rate  $n$ , no technological progress and an initial capital stock  $K(0)$ , an equilibrium path is a sequence of capital stocks, labor, output levels, consumption levels, wages and rental rates

$[K(t), L(t), Y(t), C(t), w(t), R(t)]_{t=0}^{\infty}$  such that  $L(t)$  satisfies (11),  $k(t) \equiv K(t) / L(t)$  satisfies (12),  $Y(t)$  is given by the aggregate production function,  $C(t)$  is given by (3), and  $w(t)$  and  $R(t)$  are given by (4).

- As before, *steady-state* equilibrium involves  $k(t)$  remaining constant at some level  $k^*$ .

# Steady State With Population Growth

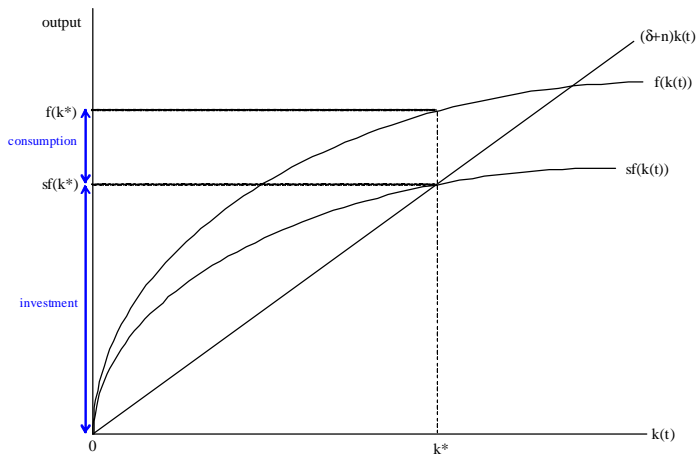


Figure: Investment and consumption in the steady-state equilibrium with population growth.

# Steady State of the Solow Model in Continuous Time

- Equilibrium path (12) has a unique *steady state* at  $k^*$ , which is given by a slight modification of (2) above:

$$\frac{f(k^*)}{k^*} = \frac{n + \delta}{s}. \quad (13)$$

**Proposition** Consider the basic Solow growth model in continuous time and suppose that Assumptions 1 and 2 hold. Then there exists a unique steady state equilibrium where the capital-labor ratio is equal to  $k^* \in (0, \infty)$  and is given by (13), per capita output is given by

$$y^* = f(k^*)$$

and per capita consumption is given by

$$c^* = (1 - s) f(k^*).$$

## Steady State of the Solow Model in Continuous Time II

- Moreover, again defining  $f(k) = a\tilde{f}(k)$ , we obtain:

**Proposition** Suppose Assumptions 1 and 2 hold and  $f(k) = a\tilde{f}(k)$ .

Denote the steady-state equilibrium level of the capital-labor ratio by  $k^*(a, s, \delta, n)$  and the steady-state level of output by  $y^*(a, s, \delta, n)$  when the underlying parameters are given by  $a$ ,  $s$  and  $\delta$ . Then we have

$$\frac{\partial k^*(\cdot)}{\partial a} > 0, \frac{\partial k^*(\cdot)}{\partial s} > 0, \frac{\partial k^*(\cdot)}{\partial \delta} < 0 \text{ and } \frac{\partial k^*(\cdot)}{\partial n} < 0$$

$$\frac{\partial y^*(\cdot)}{\partial a} > 0, \frac{\partial y^*(\cdot)}{\partial s} > 0, \frac{\partial y^*(\cdot)}{\partial \delta} < 0 \text{ and } \frac{\partial y^*(\cdot)}{\partial n} < 0.$$

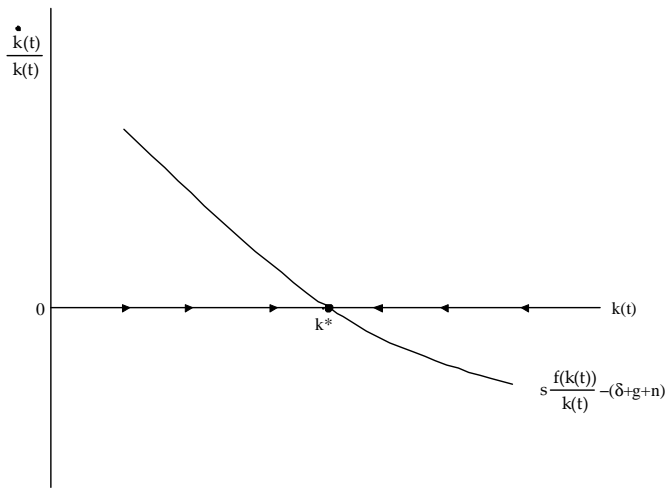
- New result is higher  $n$ , also reduces the capital-labor ratio and output per capita.
  - means there is more labor to use capital, which only accumulates slowly, thus the equilibrium capital-labor ratio ends up lower.

# Transitional Dynamics in the Continuous Time Solow Model I

## Simple Result about Stability In Continuous Time Model

- Let  $g : \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable function and suppose that there exists a unique  $x^*$  such that  $g(x^*) = 0$ . Moreover, suppose  $g(x) < 0$  for all  $x > x^*$  and  $g(x) > 0$  for all  $x < x^*$ . Then the steady state of the nonlinear differential equation  $\dot{x}(t) = g(x(t))$ ,  $x^*$ , is globally asymptotically stable, i.e., starting with any  $x(0)$ ,  $x(t) \rightarrow x^*$ .

# Simple Result in Figure



# Transitional Dynamics in the Continuous Time Solow Model II

**Proposition** Suppose that Assumptions 1 and 2 hold, then the basic Solow growth model in continuous time with constant population growth and no technological change is globally asymptotically stable, and starting from any  $k(0) > 0$ ,  $k(t) \rightarrow k^*$ .

- **Proof:** Follows immediately from the Theorem above by noting whenever  $k < k^*$ ,  $sf(k) - (n + \delta)k > 0$  and whenever  $k > k^*$ ,  $sf(k) - (n + \delta)k < 0$ .
- Figure: plots the right-hand side of (12) and makes it clear that whenever  $k < k^*$ ,  $\dot{k} > 0$  and whenever  $k > k^*$ ,  $\dot{k} < 0$ , so  $k$  monotonically converges to  $k^*$ .

# Dynamics with Cobb-Douglas Production Function I

- Return to the Cobb-Douglas Example

$$F[K, L, A] = AK^\alpha L^{1-\alpha} \text{ with } 0 < \alpha < 1.$$

- Special, mainly because elasticity of substitution between capital and labor is 1.
- Recall for a homothetic production function  $F(K, L)$ , the elasticity of substitution is

$$\sigma \equiv - \left[ \frac{\partial \ln(F_K/F_L)}{\partial \ln(K/L)} \right]^{-1}, \quad (14)$$

- $F$  is required to be homothetic, so that  $F_K/F_L$  is only a function of  $K/L$ .
- For the Cobb-Douglas production function  $F_K/F_L = (\alpha/(1-\alpha)) \cdot (L/K)$ , thus  $\sigma = 1$ .

## Dynamics with Cobb-Douglas Production Function II

- Thus when the production function is Cobb-Douglas and factor markets are competitive, equilibrium factor shares will be constant:

$$\begin{aligned}
 \alpha_K(t) &= \frac{R(t)K(t)}{Y(t)} \\
 &= \frac{F_K(K(t), L(t))K(t)}{Y(t)} \\
 &= \frac{\alpha A [K(t)]^{\alpha-1} [L(t)]^{1-\alpha} K(t)}{A [K(t)]^\alpha [L(t)]^{1-\alpha}} \\
 &= \alpha.
 \end{aligned}$$

- Similarly, the share of labor is  $\alpha_L(t) = 1 - \alpha$ .
- Reason: with  $\sigma = 1$ , as capital increases, its marginal product decreases proportionally, leaving the capital share constant.

## Dynamics with Cobb-Douglas Production Function III

- Per capita production function takes the form  $f(k) = Ak^\alpha$ , so the steady state is given again as

$$A(k^*)^{\alpha-1} = \frac{n + \delta}{s}$$

or

$$k^* = \left( \frac{sA}{n + \delta} \right)^{\frac{1}{1-\alpha}},$$

- $k^*$  is increasing in  $s$  and  $A$  and decreasing in  $n$  and  $\delta$ .
- In addition,  $k^*$  is increasing in  $\alpha$ : higher  $\alpha$  implies less diminishing returns to capital.
- Transitional dynamics are also straightforward in this case:

$$\dot{k}(t) = sA[k(t)]^\alpha - (n + \delta)k(t)$$

with initial condition  $k(0)$ .

# Dynamics with Cobb-Douglas Production Function IV

- To solve this equation, let  $x(t) \equiv k(t)^{1-\alpha}$ ,

$$\dot{x}(t) = (1 - \alpha) sA - (1 - \alpha) (n + \delta) x(t),$$

- General solution

$$x(t) = \frac{sA}{n + \delta} + \left[ x(0) - \frac{sA}{n + \delta} \right] \exp(- (1 - \alpha) (n + \delta) t).$$

- In terms of the capital-labor ratio

$$k(t) = \left\{ \frac{sA}{n + \delta} + \left[ [k(0)]^{1-\alpha} - \frac{sA}{\delta} \right] \exp(- (1 - \alpha) (n + \delta) t) \right\}^{\frac{1}{1-\alpha}}.$$

# Dynamics with Cobb-Douglas Production Function V

- This solution illustrates:
  - starting from any  $k(0)$ ,  $k(t) \rightarrow k^* = (sA / (n + \delta))^{1/(1-\alpha)}$ , and rate of adjustment is related to  $(1 - \alpha)(n + \delta)$ ,
  - more specifically, gap between  $k(0)$  and its steady-state value is closed at the exponential rate  $(1 - \alpha)(n + \delta)$ .
- Intuitive:
  - higher  $\alpha$ , less diminishing returns, slows down rate at which marginal and average product of capital declines, reduces rate of adjustment to steady state.
  - smaller  $\delta$  and smaller  $n$ : slow down the adjustment of capital per worker and thus the rate of transitional dynamics.

# Constant Elasticity of Substitution Production Function I

- Imposes a constant elasticity,  $\sigma$ , not necessarily equal to 1.
- Consider a vector-valued index of technology  $\mathbf{A}(t) = (A_H(t), A_K(t), A_L(t))$ .
- CES production function can be written as

$$\begin{aligned} Y(t) &= F[K(t), L(t), \mathbf{A}(t)] \\ &\equiv A_H(t) \left[ \gamma (A_K(t) K(t))^{\frac{\sigma-1}{\sigma}} + (1-\gamma) (A_L(t) L(t))^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} \end{aligned}$$

- $A_H(t) > 0$ ,  $A_K(t) > 0$  and  $A_L(t) > 0$  are three different types of technological change
- $\gamma \in (0, 1)$  is a distribution parameter,

# Constant Elasticity of Substitution Production Function II

- $\sigma \in [0, \infty]$  is the elasticity of substitution: easy to verify that

$$\frac{F_K}{F_L} = \frac{\gamma A_K(t)^{\frac{\sigma-1}{\sigma}} K(t)^{-\frac{1}{\sigma}}}{(1-\gamma) A_L(t)^{\frac{\sigma-1}{\sigma}} L(t)^{-\frac{1}{\sigma}}}$$

- Thus, indeed have

$$\sigma = - \left[ \frac{\partial \ln(F_K/F_L)}{\partial \ln(K/L)} \right]^{-1}.$$

# Constant Elasticity of Substitution Production Function III

- As  $\sigma \rightarrow 1$ , the CES production function converges to the Cobb-Douglas

$$Y(t) = A_H(t) (A_K(t))^\gamma (A_L(t))^{1-\gamma} (K(t))^\gamma (L(t))^{1-\gamma}$$

- As  $\sigma \rightarrow \infty$ , the CES production function becomes linear, i.e.

$$Y(t) = \gamma A_H(t) A_K(t) K(t) + (1 - \gamma) A_H(t) A_L(t) L(t).$$

- Finally, as  $\sigma \rightarrow 0$ , the CES production function converges to the Leontief production function with no substitution between factors,

$$Y(t) = A_H(t) \min \{ \gamma A_K(t) K(t); (1 - \gamma) A_L(t) L(t) \}.$$

- Leontief production function: if  $\gamma A_K(t) K(t) \neq (1 - \gamma) A_L(t) L(t)$ , either capital or labor will be partially “idle”.

# A First Look at Sustained Growth I

- Cobb-Douglas already showed that when  $\alpha$  is close to 1, adjustment to steady-state level can be very slow.
- Simplest model of sustained growth essentially takes  $\alpha = 1$  in terms of the Cobb-Douglas production function above.
- Relax Assumptions 1 and 2 and suppose

$$F [K (t) , L (t) , A (t)] = AK (t) , \quad (15)$$

where  $A > 0$  is a constant.

- So-called “AK” model, and in its simplest form output does not even depend on labor.
- Results we would like to highlight apply with more general constant returns to scale production functions,

$$F [K (t) , L (t) , A (t)] = AK (t) + BL (t) , \quad (16)$$

# A First Look at Sustained Growth II

- Assume population grows at  $n$  as before (cfr. equation (11)).
- Combining with the production function (15),

$$\frac{\dot{k}(t)}{k(t)} = sA - \delta - n.$$

- Therefore, if  $sA - \delta - n > 0$ , there will be sustained growth in the capital-labor ratio.
- From (15), this implies that there will be sustained growth in output per capita as well.

## A First Look at Sustained Growth III

**Proposition** Consider the Solow growth model with the production function (15) and suppose that  $sA - \delta - n > 0$ . Then in equilibrium, there is sustained growth of output per capita at the rate  $sA - \delta - n$ . In particular, starting with a capital-labor ratio  $k(0) > 0$ , the economy has

$$k(t) = \exp((sA - \delta - n)t) k(0)$$

and

$$y(t) = \exp((sA - \delta - n)t) A k(0).$$

- Note no transitional dynamics.

# Sustained Growth in Figure

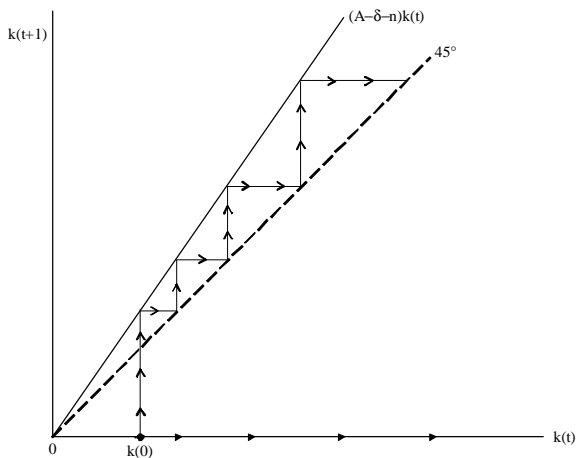


Figure: Sustained growth with the linear AK technology with  $sA - \delta - n > 0$ .

# A First Look at Sustained Growth IV

- Unattractive features:
  - ① Knife-edge case, requires the production function to be ultimately linear in the capital stock.
  - ② Implies that as time goes by the share of national income accruing to capital will increase towards 1.
  - ③ Technological progress seems to be a major (perhaps the most major) factor in understanding the process of economic growth.

# Balanced Growth I

- Production function  $F [K (t) , L (t) , A (t)]$  is too general.
- May not have *balanced growth*, i.e. a path of the economy consistent with the *Kaldor facts* (Kaldor, 1963).
- Kaldor facts:
  - while output per capita increases, the capital-output ratio, the interest rate, and the distribution of income between capital and labor remain roughly constant.

# Historical Factor Shares

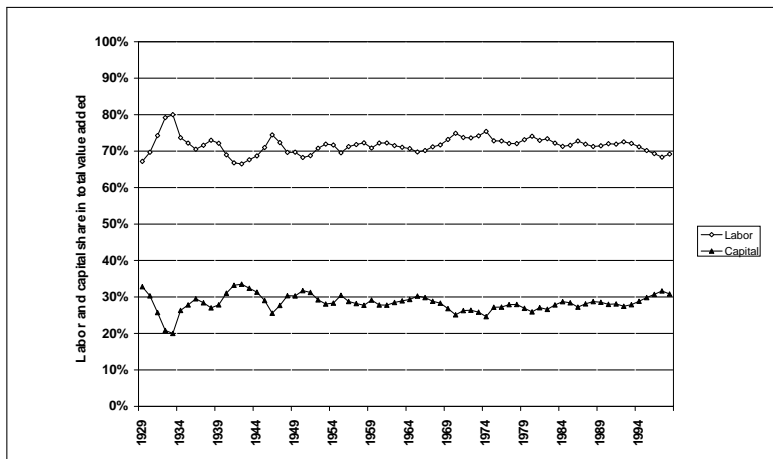


Figure: Capital and Labor Share in the U.S. GDP.

## Balanced Growth II

- Note capital share in national income is about  $1/3$ , while the labor share is about  $2/3$ .
- Ignoring land, not a major factor of production.
- But in poor countries land is a major factor of production.
- This pattern often makes economists choose  $AK^{1/3}L^{2/3}$ .
- Main advantage from our point of view is that balanced growth is the same as a steady-state in transformed variables
  - i.e., we will again have  $\dot{k} = 0$ , but the definition of  $k$  will change.
- But important to bear in mind that growth has many non-balanced features.
  - e.g., the share of different sectors changes systematically.

# Types of Neutral Technological Progress I

- For some constant returns to scale function  $\tilde{F}$ :

- *Hicks-neutral* technological progress:

$$\tilde{F}[K(t), L(t), A(t)] = A(t) F[K(t), L(t)],$$

- Relabeling of the isoquants (without any change in their shape) of the function  $\tilde{F}[K(t), L(t), A(t)]$  in the  $L$ - $K$  space.
- *Solow-neutral* technological progress,

$$\tilde{F}[K(t), L(t), A(t)] = F[A(t)K(t), L(t)].$$

- Capital-augmenting progress: isoquants shifting with technological progress in a way that they have constant slope at a given labor-output ratio.
- *Harrod-neutral* technological progress,

$$\tilde{F}[K(t), L(t), A(t)] = F[K(t), A(t)L(t)].$$

- Increases output as if the economy had more labor: slope of the isoquants are constant along rays with constant capital-output ratio.

# Isoquants with Neutral Technological Progress

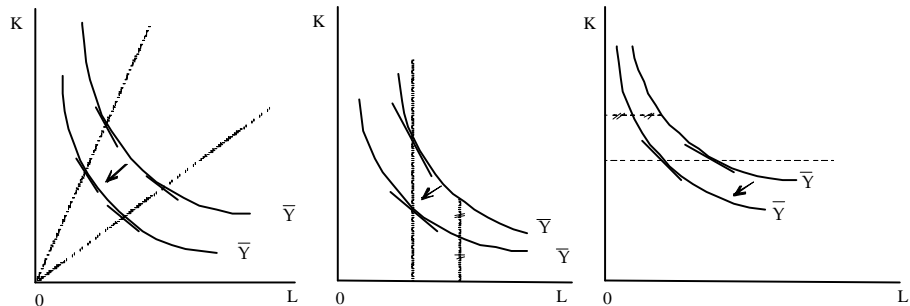


Figure: Hicks-neutral, Solow-neutral and Harrod-neutral shifts in isoquants.

## Types of Neutral Technological Progress II

- Could also have a vector valued index of technology  $\mathbf{A}(t) = (A_H(t), A_K(t), A_L(t))$  and a production function

$$\tilde{F}[K(t), L(t), \mathbf{A}(t)] = A_H(t) F[A_K(t) K(t), A_L(t) L(t)], \quad (17)$$

- Nests the constant elasticity of substitution production function introduced in the Example above.
- But even (17) is a restriction on the form of technological progress,  $A(t)$  could modify the entire production function.
- Balanced growth necessitates that all technological progress be labor augmenting or Harrod-neutral.

# Uzawa's Theorem I

- Focus on continuous time models.
- Key elements of balanced growth: constancy of factor shares and of the capital-output ratio,  $K(t) / Y(t)$ .
- By factor shares, we mean

$$\alpha_L(t) \equiv \frac{w(t) L(t)}{Y(t)} \quad \text{and} \quad \alpha_K(t) \equiv \frac{R(t) K(t)}{Y(t)}.$$

- By Assumption 1 and Euler Theorem  $\alpha_L(t) + \alpha_K(t) = 1$ .

# Uzawa's Theorem II

## Theorem

**(Uzawa I)** Suppose  $L(t) = \exp(nt) L(0)$ ,

$$Y(t) = \tilde{F}(K(t), L(t), \tilde{A}(t)),$$

$\dot{K}(t) = Y(t) - C(t) - \delta K(t)$ , and  $\tilde{F}$  is CRS in  $K$  and  $L$ .

Suppose for  $\tau < \infty$ ,  $\dot{Y}(t)/Y(t) = g_Y > 0$ ,  $\dot{K}(t)/K(t) = g_K > 0$  and  $\dot{C}(t)/C(t) = g_C > 0$ . Then,

- 1  $g_Y = g_K = g_C$ ; and
- 2 for any  $t \geq \tau$ ,  $\tilde{F}$  can be represented as

$$Y(t) = F(K(t), A(t)L(t)),$$

where  $A(t) \in \mathbb{R}_+$ ,  $F: \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$  is homogeneous of degree 1, and

$$\dot{A}(t)/A(t) = g = g_Y - n.$$

# Proof of Uzawa's Theorem I

- By hypothesis,  $Y(t) = \exp(g_Y(t - \tau)) Y(\tau)$ ,  $K(t) = \exp(g_K(t - \tau)) K(\tau)$  and  $L(t) = \exp(n(t - \tau)) L(\tau)$  for some  $\tau < \infty$ .
- Since for  $t \geq \tau$ ,  $\dot{K}(t) = g_K K(t) = I(t) - C(t) - \delta K(t)$ , we have

$$(g_K + \delta) K(t) = Y(t) - C(t).$$

- Then,

$$\begin{aligned} (g_K + \delta) K(\tau) &= \exp((g_Y - g_K)(t - \tau)) Y(\tau) \\ &\quad - \exp((g_C - g_K)(t - \tau)) C(\tau) \end{aligned}$$

for all  $t \geq \tau$ .

## Proof of Uzawa's Theorem II

- Differentiating with respect to time

$$0 = (g_Y - g_K) \exp((g_Y - g_K)(t - \tau)) Y(\tau) - (g_C - g_K) \exp((g_C - g_K)(t - \tau)) C(\tau)$$

for all  $t \geq \tau$ .

- This equation can hold for all  $t \geq \tau$ 
  - if  $g_Y = g_C$  and  $Y(\tau) = C(\tau)$ , which is not possible, since  $g_K > 0$ .
  - or if  $g_Y = g_K$  and  $C(\tau) = 0$ , which is not possible, since  $g_C > 0$  and  $C(\tau) > 0$ .
  - or if  $g_Y = g_K = g_C$ , which must thus be the case.
- Therefore,  $g_Y = g_K = g_C$  as claimed in the first part of the theorem.

## Proof of Uzawa's Theorem III

- Next, the aggregate production function for time  $\tau' \geq \tau$  and any  $t \geq \tau$  can be written as

$$\begin{aligned} & \exp(-g_Y(t - \tau')) Y(t) \\ &= \tilde{F} [\exp(-g_K(t - \tau')) K(t), \exp(-n(t - \tau')) L(t), \tilde{A}(\tau')] \end{aligned}$$

- Multiplying both sides by  $\exp(g_Y(t - \tau'))$  and using the constant returns to scale property of  $F$ , we obtain

$$Y(t) = \tilde{F} \left[ e^{(t-\tau')(g_Y - g_K)} K(t), e^{(t-\tau')(g_Y - n)} L(t), \tilde{A}(\tau') \right].$$

- From part 1,  $g_Y = g_K$ , therefore

$$Y(t) = \tilde{F} [K(t), \exp((t - \tau')(g_Y - n)) L(t), \tilde{A}(\tau')].$$

## Proof of Uzawa's Theorem IV

- Moreover, this equation is true for  $t \geq \tau$  regardless of  $\tau'$ , thus

$$\begin{aligned} Y(t) &= F[K(t), \exp((g_Y - n)t) L(t)], \\ &= F[K(t), A(t) L(t)], \end{aligned}$$

with

$$\frac{\dot{A}(t)}{A(t)} = g_Y - n$$

establishing the second part of the theorem.

# Implications of Uzawa's Theorem

**Corollary** Under the assumptions of Uzawa Theorem, after time  $\tau$  technological progress can be represented as Harrod neutral (purely labor augmenting).

- Remarkable feature: stated and proved without any reference to equilibrium behavior or market clearing.
- Also, contrary to Uzawa's original theorem, not stated for a balanced growth path but only for an asymptotic path with constant rates of output, capital and consumption growth.
- **But**, not as general as it seems;
  - the theorem gives only one representation.

# Stronger Theorem

## Theorem

**(Uzawa's Theorem II)** Suppose that all of the hypothesis in Uzawa's Theorem are satisfied, so that  $\tilde{F} : \mathbb{R}_+^2 \times \mathcal{A} \rightarrow \mathbb{R}_+$  has a representation of the form  $F(K(t), A(t)L(t))$  with  $A(t) \in \mathbb{R}_+$  and  $\dot{A}(t)/A(t) = g = g_Y - n$ . In addition, suppose that factor markets are competitive and that for all  $t \geq T$ , the rental rate satisfies  $R(t) = R^*$  (or equivalently,  $\alpha_K(t) = \alpha_K^*$ ). Then, denoting the partial derivatives of  $\tilde{F}$  and  $F$  with respect to their first two arguments by  $\tilde{F}_K, \tilde{F}_L, F_K$  and  $F_L$ , we have

$$\begin{aligned} \tilde{F}_K(K(t), L(t), \tilde{A}(t)) &= F_K(K(t), A(t)L(t)) \text{ and} & (18) \\ \tilde{F}_L(K(t), L(t), \tilde{A}(t)) &= A(t)F_L(K(t), A(t)L(t)). \end{aligned}$$

Moreover, if (18) holds and factor markets are competitive, then  $R(t) = R^*$  (and  $\alpha_K(t) = \alpha_K^*$ ) for all  $t \geq T$ .

# Intuition

- Suppose the labor-augmenting representation of the aggregate production function applies.
- Then note that with competitive factor markets, as  $t \geq \tau$ ,

$$\begin{aligned}
 \alpha_K(t) &\equiv \frac{R(t) K(t)}{Y(t)} \\
 &= \frac{K(t)}{Y(t)} \frac{\partial F[K(t), A(t)L(t)]}{\partial K(t)} \\
 &= \alpha_K^*,
 \end{aligned}$$

- Second line uses the definition of the rental rate of capital in a competitive market
- Third line uses that  $g_Y = g_K$  and  $g_K = g + n$  from Uzawa Theorem and that  $F$  exhibits constant returns to scale so its derivative is homogeneous of degree 0.

## Intuition for the Uzawa's Theorems

- We assumed the economy features capital accumulation in the sense that  $g_K > 0$ .
- From the aggregate resource constraint, this is only possible if output and capital grow at the same rate.
- Either this growth rate is equal to  $n$  and there is no technological change (i.e., proposition applies with  $g = 0$ ), or the economy exhibits growth of per capita income and capital-labor ratio.
- The latter case creates an asymmetry between capital and labor: capital is accumulating faster than labor.
- Constancy of growth requires technological change to make up for this asymmetry
- But this intuition does not provide a reason for why technology should take labor-augmenting (Harrod-neutral) form.
- But if technology did not take this form, an asymptotic path with constant growth rates would not be possible.

# Interpretation

- Distressing result:
  - Balanced growth is only possible under a very stringent assumption.
  - Provides no reason why technological change should take this form.
- But when technology is endogenous, intuition above also works to make technology endogenously more labor-augmenting than capital augmenting.
- Not only requires labor augmenting asymptotically, i.e., along the balanced growth path.
- This is the pattern that certain classes of endogenous-technology models will generate.

## Implications for Modeling of Growth

- Does not require  $Y(t) = F[K(t), A(t)L(t)]$ , but only that it has a representation of the form  $Y(t) = F[K(t), A(t)L(t)]$ .
- Allows one important exception. If,

$$Y(t) = [A_K(t)K(t)]^\alpha [A_L(t)L(t)]^{1-\alpha},$$

then both  $A_K(t)$  and  $A_L(t)$  could grow asymptotically, while maintaining balanced growth.

- Because we can define  $A(t) = [A_K(t)]^{\alpha/(1-\alpha)} A_L(t)$  and the production function can be represented as

$$Y(t) = [K(t)]^\alpha [A(t)L(t)]^{1-\alpha}.$$

- Differences between labor-augmenting and capital-augmenting (and other forms) of technological progress matter when the elasticity of substitution between capital and labor is not equal to 1.

## Further Intuition

- Suppose the production function takes the special form  $F [A_K (t) K (t) , A_L (t) L (t)]$ .
- The stronger theorem implies that factor shares will be constant.
- Given constant returns to scale, this can only be the case when  $A_K (t) K (t)$  and  $A_L (t) L (t)$  grow at the same rate.
- The fact that the capital-output ratio is constant in steady state (or the fact that capital accumulates) implies that  $K (t)$  must grow at the same rate as  $A_L (t) L (t)$ .
- Thus balanced growth can only be possible if  $A_K (t)$  is asymptotically constant.

# The Solow Growth Model with Technological Progress: Continuous Time I

- From Uzawa Theorem, production function must admit representation of the form

$$Y(t) = F[K(t), A(t)L(t)],$$

- Moreover, suppose

$$\frac{\dot{A}(t)}{A(t)} = g, \quad (19)$$

$$\frac{\dot{L}(t)}{L(t)} = n.$$

- Again using the constant saving rate

$$\dot{K}(t) = sF[K(t), A(t)L(t)] - \delta K(t). \quad (20)$$

# The Solow Growth Model with Technological Progress: Continuous Time II

- Now define  $k(t)$  as the *effective capital-labor* ratio, i.e.,

$$k(t) \equiv \frac{K(t)}{A(t)L(t)}. \quad (21)$$

- Slight but useful abuse of notation.
- Differentiating this expression with respect to time,

$$\frac{\dot{k}(t)}{k(t)} = \frac{\dot{K}(t)}{K(t)} - g - n. \quad (22)$$

- Output per unit of effective labor can be written as

$$\begin{aligned} \hat{y}(t) &\equiv \frac{Y(t)}{A(t)L(t)} = F \left[ \frac{K(t)}{A(t)L(t)}, 1 \right] \\ &\equiv f(k(t)). \end{aligned}$$

# The Solow Growth Model with Technological Progress: Continuous Time III

- Income per capita is  $y(t) \equiv Y(t) / L(t)$ , i.e.,

$$\begin{aligned}y(t) &= A(t) \hat{y}(t) \\ &= A(t) f(k(t)).\end{aligned}\tag{23}$$

- Clearly if  $\hat{y}(t)$  is constant, income per capita,  $y(t)$ , will grow over time, since  $A(t)$  is growing.
- Thus should not look for “steady states” where income per capita is constant, but for *balanced growth paths*, where income per capita grows at a constant rate.
- Some transformed variables such as  $\hat{y}(t)$  or  $k(t)$  in (22) remain constant.
- Thus balanced growth paths can be thought of as steady states of a transformed model.

# The Solow Growth Model with Technological Progress: Continuous Time IV

- Hence use the terms “steady state” and balanced growth path interchangeably.
- Substituting for  $\dot{K}(t)$  from (20) into (22):

$$\frac{\dot{k}(t)}{k(t)} = \frac{sF[K(t), A(t)L(t)]}{K(t)} - (\delta + g + n).$$

- Now using (21),

$$\frac{\dot{k}(t)}{k(t)} = \frac{sf(k(t))}{k(t)} - (\delta + g + n), \quad (24)$$

- Only difference is the presence of  $g$ :  $k$  is no longer the capital-labor ratio but the *effective* capital-labor ratio.

# The Solow Growth Model with Technological Progress: Continuous Time V

**Proposition** Consider the basic Solow growth model in continuous time, with Harrod-neutral technological progress at the rate  $g$  and population growth at the rate  $n$ . Suppose that Assumptions 1 and 2 hold, and define the effective capital-labor ratio as in (21). Then there exists a unique steady state (balanced growth path) equilibrium where the effective capital-labor ratio is equal to  $k^* \in (0, \infty)$  and is given by

$$\frac{f(k^*)}{k^*} = \frac{\delta + g + n}{s}. \quad (25)$$

Per capita output and consumption grow at the rate  $g$ .

# The Solow Growth Model with Technological Progress: Continuous Time VI

- Equation (25), emphasizes that now total savings,  $sf(k)$ , are used for replenishing the capital stock for three distinct reasons:
  - 1 depreciation at the rate  $\delta$ .
  - 2 population growth at the rate  $n$ , which reduces capital per worker.
  - 3 Harrod-neutral technological progress at the rate  $g$ .
- Now replenishment of effective capital-labor ratio requires investments to be equal to  $(\delta + g + n)k$ .

# The Solow Growth Model with Technological Progress: Continuous Time VII

**Proposition** Suppose Assumptions 1 and 2 hold and let  $A(0)$  be the initial level of technology. Denote the balanced growth path level of effective capital-labor ratio by  $k^*(A(0), s, \delta, n)$  and the level of output per capita by  $y^*(A(0), s, \delta, n, t)$ . Then

$$\frac{\partial k^*(A(0), s, \delta, n)}{\partial A(0)} = 0, \quad \frac{\partial k^*(A(0), s, \delta, n)}{\partial s} > 0,$$

$$\frac{\partial k^*(A(0), s, \delta, n)}{\partial n} < 0 \text{ and } \frac{\partial k^*(A(0), s, \delta, n)}{\partial \delta} < 0,$$

and also

$$\frac{\partial y^*(A(0), s, \delta, n, t)}{\partial A(0)} > 0, \quad \frac{\partial y^*(A(0), s, \delta, n, t)}{\partial s} > 0,$$

$$\frac{\partial y^*(A(0), s, \delta, n, t)}{\partial n} < 0 \text{ and } \frac{\partial y^*(A(0), s, \delta, n, t)}{\partial \delta} < 0,$$

for each  $t$ .

# The Solow Growth Model with Technological Progress: Continuous Time VIII

**Proposition** Suppose that Assumptions 1 and 2 hold, then the Solow growth model with Harrod-neutral technological progress and population growth in continuous time is asymptotically stable, i.e., starting from any  $k(0) > 0$ , the effective capital-labor ratio converges to a steady-state value  $k^*$  ( $k(t) \rightarrow k^*$ ).

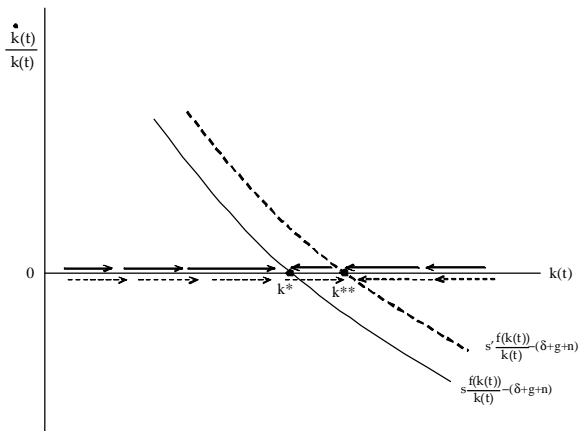
- Now model generates growth in output per capita, but entirely *exogenously*.

# Comparative Dynamics I

- Comparative dynamics: dynamic response of an economy to a change in its parameters or to shocks.
- Different from comparative statics in Propositions above in that we are interested in the entire path of adjustment of the economy following the shock or changing parameter.
- For brevity we will focus on the continuous time economy.
- Recall

$$\dot{k}(t) / k(t) = sf(k(t)) / k(t) - (\delta + g + n)$$

# Comparative Dynamics in Figure



**Figure:** Dynamics following an increase in the savings rate from  $s$  to  $s'$ . The solid arrows show the dynamics for the initial steady state, while the dashed arrows

# Comparative Dynamics II

- One-time, unanticipated, permanent increase in the saving rate from  $s$  to  $s'$ .
  - Shifts curve to the right as shown by the dotted line, with a new intersection with the horizontal axis,  $k^{**}$ .
  - Arrows on the horizontal axis show how the effective capital-labor ratio adjusts gradually to  $k^{**}$ .
  - Immediately, the capital stock remains unchanged (since it is a *state* variable).
  - After this point, it follows the dashed arrows on the horizontal axis.
- $s$  changes in unanticipated manner at  $t = t'$ , but will be reversed back to its original value at some known future date  $t = t'' > t'$ .
  - Starting at  $t'$ , the economy follows the rightwards arrows until  $t'$ .
  - After  $t''$ , the original steady state of the differential equation applies and leftwards arrows become effective.
  - From  $t''$  onwards, economy gradually returns back to its original balanced growth equilibrium,  $k^*$ .

# Conclusions

- Simple and tractable framework, which allows us to discuss capital accumulation and the implications of technological progress.
- Solow model shows us that if there is no technological progress, and as long as we are not in the *AK* world, there will be no sustained growth.
- Generate per capita output growth, but only exogenously: technological progress is a blackbox.
- Capital accumulation: determined by the saving rate, the depreciation rate and the rate of population growth. All are exogenous.
- Need to dig deeper and understand what lies in these black boxes.